STEP 2021, P3, Q12 - Solution (7 pages; 10/7/23)

(i) **1st Part**

By symmetry, all possible numbers rolled by Player 2 are equally likely, whatever the state of X_{12}, and so knowledge of the state of X_{12} does not affect knowledge of the state of X_{23}. Therefore X_{23} is independent of X_{12}, and vice-versa.

2nd Part

Let X be the total score, so that $X = \sum_{1\leq i<j\leq n} X_{ij}$

and $E(X) = \sum_{1\leq i<j\leq n} E(X_{ij})$

The number of different X_{ij} , where $1 \leq i < j \leq n$, is

$(n-1)+(n-2)+\cdots+0$ (as there are $n-1$ possible numbers greater than 1 etc)

$= \frac{1}{2}(n-1)n$ [or just ${}^nC_2$]

and so $E(X) = \frac{1}{2}(n-1)n\, E(X_{12})$, by symmetry; and

$E(X_{12}) = P(Player\ 2\ rolls\ the\ same\ number\ as\ Player\ 1) \times\ 1$

$+P(Player\ 2\ rolls\ a\ different\ number\ to\ Player\ 1) \times\ 0$

$= \frac{1}{6}$

and hence $E(X) = \frac{1}{12}(n-1)n$

3rd Part

As well as X_{12} & X_{23} being independent, clearly X_{12} & X_{34} etc will be independent, so that all X_{ij} are independent of each other.

Because of the independence of the X_{ij},

$Var(X) = \sum_{1\leq i<j\leq n} Var(X_{ij})$

$= \frac{1}{2}(n-1)n\, Var(X_{12})$, again by symmetry.

And $Var(X_{12}) = E\left({X_{12}}^2\right) - [E(X_{12})]^2$

where $E\left({X_{12}}^2\right)$

$= (Player\ 2\ rolls\ the\ same\ number\ as\ Player\ 1) \times\ 1^2$

$+P(Player\ 2\ rolls\ a\ different\ number\ to\ Player\ 1) \times\ 0^2$

$= \frac{1}{6}$,

so that $Var(X) = \frac{1}{2}(n-1)n\ [\frac{1}{6} - \left(\frac{1}{6}\right)^2]$

$= \frac{5}{72}(n-1)n$

(ii) $Var(Y_1 + \cdots + Y_m) = E[(Y_1 + \cdots + Y_m)^2] - [E(Y_1 + \cdots + Y_m)]^2$

$= \sum_{i=1}^{m} E\left({Y_i}^2\right) + 2\sum_{i<j} E\left(Y_i Y_j\right) - (\sum_{i=1}^{m} E(Y_i))^2$

$= [\sum_{i=1}^{m} E\left({Y_i}^2\right)] + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} E\left(Y_i Y_j\right)$ (as each $E(Y_i) = 0$)

as required

(iii) **1st Part**

The knowledge that $Z_{12} = 1$, for example, means that Player 2 rolled an even number, so that $Z_{23} = -1$ is not possible. So the knowledge of the state of Z_{12} can affect knowledge of the state of Z_{23}, and therefore Z_{23} is not independent of Z_{12}, and vice-versa.

2nd Part

Let Z be the total score, so that $Z = \sum_{1 \le i < j \le n} Z_{ij}$

Once again, $E(Z) = \frac{1}{2}(n-1)n\, E(Z_{12})$

And $E(Z_{12}) = Prob(Players\ 1\ \&\ 2\ both\ roll$ the same even number)$\times 1$

$+Prob(Players\ 1\ \&\ 2\ both\ roll$ the same odd number)$\times (-1)$

$+ 0$

$= Prob(Player\ 1\ rolls\ an\ even\ number)$

$\times Prob(Player\ 2\ rolls\ the\ same\ number)$

$-Prob(Player\ 1\ rolls\ an\ odd\ number)$

$\times Prob(Player\ 2\ rolls\ the\ same\ number)$

$= \frac{1}{2} . \frac{1}{6} - \frac{1}{2} . \frac{1}{6} = 0$

So $E(Z) = 0$.

3rd Part

$VarZ = [\sum_{1 \le i < j \le n} E({Z_{ij}}^2)]$

$+2 \sum E\left(Z_{ij} Z_{kl}\right)$, where $i < j$ & $k < l$, and eg $Z_{12}Z_{34}$ and $Z_{34}Z_{12}$ count as the same item [the multiple of 2 already allows for this]

Now, $E({Z_{ij}}^2)$

$= Prob(Player\ i\ rolls\ an\ even\ number)$

$\times Prob(Player\ j\ rolls\ the\ same\ number) \times 1^2$

$+Prob(Player\ i\ rolls\ an\ odd\ number)$

$\times Prob(Player\ j\ rolls\ the\ same\ number) \times (-1)^2$

$= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{6}$

and, as in the 2nd Part of (i), there are $\frac{1}{2}(n-1)n$ ways of choosing i & j, so that $= \sum_{1 \le i < j \le n} E({Z_{ij}}^2) = \frac{1}{12}(n-1)n$

For $E(Z_{ij}Z_{kl})$ (where $i < j, k < l$; and eg $Z_{12}Z_{34}$ and $Z_{34}Z_{12}$ count as the same item):

When there are no numbers in common between i, j, k & l,

Z_{ij} & Z_{kl} are independent,

and so $E(Z_{ij}Z_{kl}) = E(Z_{ij})E(Z_{kl}) = 0 \times 0$

Other cases will fall into one of the following categories:

Category A: eg $Z_{47}Z_{49}$ (with $7 < 9$, as $Z_{47}Z_{49}$ & $Z_{49}Z_{47}$ count as the same item)

Category B: eg $Z_{47}Z_{24}$ or $Z_{24}Z_{47}$

Category C: eg $Z_{47}Z_{57}$ (with $4 < 5$)

[See note below.]

For Category A, $Z_{47}Z_{49}$ (eg) will be non-zero when

the 4th player has an even number, the 7th player has the same number & the 9th player has the same number as well,

or when the 4th player has an odd number, the 7th player has the same number & the 9th player has the same number as well.

So $E(Z_{47}Z_{49}) = \frac{1}{2}\times\frac{1}{6}\times\frac{1}{6}\times 1^2 + \frac{1}{2}\times\frac{1}{6}\times\frac{1}{6}\times(-1)^2 = \frac{1}{36}$

Similar reasoning applies to Categories B and C.

To count the number of items in Category A:

For items of the form $Z_{1j}Z_{1l}$ (where $j < l$):

${}^{n-1}C_2 = \frac{1}{2}(n-1)(n-2)$

For items of the form $Z_{2j}Z_{2l}$ (where $j < l$):

${}^{n-2}C_2 = \frac{1}{2}(n-2)(n-3)$

... For items of the form $Z_{(n-2)j}Z_{(n-2)l}$ (where $j < l$): 1

So total number of items in Category A is

$\frac{1}{2}\sum_{r=1}^{n-2}(n-r)(n-r-1)$

Writing $k = n-r-1$, this becomes

$\frac{1}{2}\sum_{k=n-2}^{1} k(k-1)$

or $\frac{1}{2}\sum_{k=1}^{n-2} k(k-1) = \frac{1}{2}.\frac{1}{6}(n-2)(n-1)(2n-3)$

$+\frac{1}{2}.\frac{1}{2}(n-2)(n-1)$

$= \frac{1}{12}(n-1)(n-2)[2n-3+3] = \frac{n}{6}(n-1)(n-2)$

To count the number of items in Category B:

Items of the form $Z_{2j}Z_{k2}$: $(n-2).1$

Items of the form $Z_{3j}Z_{k3}$: $(n-3).2$

... Items of the form $Z_{(n-1)j}Z_{k(n-1)}$: $1.(n-2)$

So total number of items in Category B is

$\sum_{r=1}^{n-2} r(n-r-1) = (n-1).\frac{1}{2}(n-2)(n-1)$

$-\frac{1}{6}(n-2)(n-1)(2n-3)$

$= \frac{1}{6}(n-1)(n-2)[3n-3-(2n-3)]$

$= \frac{n}{6}(n-1)(n-2)$

To count the number of items in Category C:

This will be the same as for Category A, with ${}^{n-1}C_2$ items of the form $Z_{in}Z_{kn}$ (with $i < k$) etc.

Thus the total number of items in Categories A, B & C (together) is

$3 \times \frac{n}{6}(n-1)(n-2)$

and each $E(Z_{ij}Z_{kl})$ for these items is $\frac{1}{36}$,

so that $VarZ = [\sum_{1 \leq i < j \leq n} E({Z_{ij}}^2)] + 2\sum E(Z_{ij}Z_{kl})$

$= \frac{1}{12}(n-1)n + 2 \times 3 \times \frac{n}{6}(n-1)(n-2)\left(\frac{1}{36}\right)$

$= \frac{n}{36}(n-1)[3+n-2] = \frac{1}{36}n(n^2-1)$, as required.

Note: The Official Solution uses the result of (ii) more directly, by writing $Y_1 = Z_{12},\ Y_2 = Z_{13}, \ldots, Y_m = Z_{(n-1)n}$, with $m = {}^nC_2$.

However, it doesn't explain the $n \times {}^{n-1}C_2$ appearing in the expression for $2\sum E(Y_iY_j)$ (or $2\sum E(Z_{ij}Z_{kl})$). This can be justified as follows:

As explained above, the only non-zero $E(Z_{ij}Z_{kl})$ items are the ones falling into the 3 categories mentioned. As an alternative to the 3 categories, these items can be classified according to which of the n possible numbers is repeated (note that it isn't possible for a number to appear more than twice, as $i < j$ & $k < l$). There are then ${}^{n-1}C_2$ ways of choosing the other two numbers, and for each choice of the repeated number and the other 2 numbers, exactly one of the categories A, B or C must occur: Suppose, for example, that the repeated number is 4. Then the other 2 numbers will either both be less than 4, or both be greater than 4, or lie on either side of 4 (corresponding to the categories C, A & B).