STEP 2021, P3, Q11 - Solution (3 pages; 6/7/23)

(i) $P(Y=n)=\int_n^{n+1}\lambda e^{-\lambda x}dx$

$=\lambda[-\frac{1}{\lambda}e^{-\lambda x}]_n^{n+1}$

$=-(e^{-\lambda(n+1)}-e^{-\lambda n})$

$=(1-e^{-\lambda})e^{-\lambda n}$, as required.

(ii) Conditioning on Y:

$P(Z<z)=\sum_{n=0}^{\infty}P(Y=n)P(Z<z|Y=n)$

and $P(Z<z|Y=n)=P(n<X<n+z|n<X<n+1)$

$=\frac{\int_n^{n+z}\lambda e^{-\lambda x}dx}{P(Y=n)}$ (where $0\le z\le 1$)

So $P(Z<z)=\sum_{n=0}^{\infty}\int_n^{n+z}\lambda e^{-\lambda x}dx$

[Alternatively, we could just have started with

$P(Z<z)=\sum_{n=0}^{\infty}P(n<X<n+z)$]

and $\int_n^{n+z}\lambda e^{-\lambda x}dx=\lambda[-\frac{1}{\lambda}e^{-\lambda x}]_n^{n+z}$

$=-(e^{-\lambda(n+z)}-e^{-\lambda n})$

$=(1-e^{-\lambda z})e^{-\lambda n}$

Hence $P(Z<z)=\sum_{n=0}^{\infty}(1-e^{-\lambda z})e^{-\lambda n}$

$=\left(1-e^{-\lambda z}\right).\frac{1}{1-e^{-\lambda}}$ (GP with 1st term 1 & common ratio $e^{-\lambda}$)

(iii) Let the pdf of z be $f_Z(z) = \frac{d}{dz}P(Z < z)$

$= \frac{1}{1-e^{-\lambda}}(\lambda e^{-\lambda z})$

Then $E(Z) = \int_0^1 \frac{z}{1-e^{-\lambda}}\left(\lambda e^{-\lambda z}\right)dz$

$= \frac{\lambda}{1-e^{-\lambda}}\{\left[z.\left(-\frac{1}{\lambda}\right)e^{-\lambda z}\right]_0^1 - \int_0^1 \left(-\frac{1}{\lambda}\right)e^{-\lambda z}dz\}$ (by Parts)

$= \frac{\lambda}{1-e^{-\lambda}}\{\left(-\frac{1}{\lambda}\right)e^{-\lambda} + \frac{1}{\lambda}\left[\left(-\frac{1}{\lambda}\right)e^{-\lambda z}\right]_0^1\}$

$= \frac{-1}{1-e^{-\lambda}}\{e^{-\lambda} + \frac{1}{\lambda}\left(e^{-\lambda} - 1\right)\}$

$= \frac{e^{\lambda}-1-\lambda}{\lambda(e^{\lambda}-1)} = \frac{1}{\lambda} - \frac{1}{e^{\lambda}-1}$

(iv) **1st Part**

$P(Y = n\ and\ z_1 < Z < z_2) = \int_{n+z_1}^{n+z_2} \lambda e^{-\lambda x}dx$

$= \lambda[-\frac{1}{\lambda}e^{-\lambda x}]_{n+z_1}^{n+z_2}$

$= -(e^{-\lambda(n+z_2)} - e^{-\lambda(n+z_1)})$

$= e^{-\lambda n}(e^{-\lambda z_1} - e^{-\lambda z_2})$

2nd Part

Result to prove:

$P(Y = n\ and\ z_1 < Z < z_2) = P(Y = n).P(z_1 < Z < z_2)$

[This can be assumed to be the condition for independence of the discrete variable Y and the continuous variable Z.]

From (i), $P(Y = n) = (1 - e^{-\lambda})e^{-\lambda n}$

And $P(z_1 < Z < z_2) = P(Z < z_2) - P(Z < z_1)$

$= [(1 - e^{-\lambda z_2}) - (1 - e^{-\lambda z_1})].\frac{1}{1-e^{-\lambda}}$, from (ii)

$= \frac{e^{-\lambda z_1} - e^{-\lambda z_2}}{1-e^{-\lambda}}$

So $P(Y = n).P(z_1 < Z < z_2) = (e^{-\lambda z_1} - e^{-\lambda z_2})e^{-\lambda n}$,

which equals $P(Y = n \; and \; z_1 < Z < z_2)$, from the 1st Part,

as required.