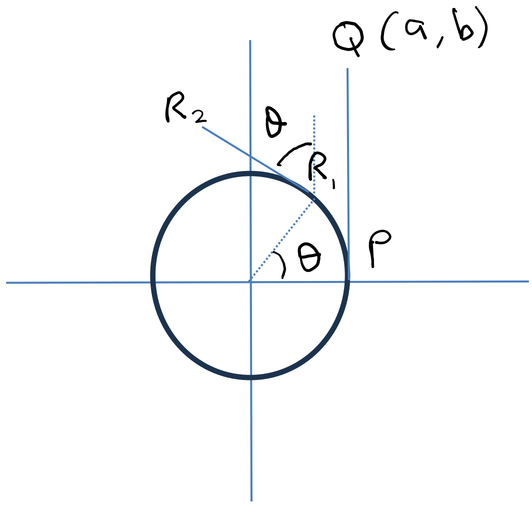
STEP 2021, P3, Q10 - Solution (4 pages; 1/7/23)

(i) 1st Part



Referring to the diagrams, the arc length from P to R_1 is $a\theta$, and R_1 has coordinates $(acos\theta, asin\theta)$.

Then, noting that the angle that R_1R_2 makes with the upward vertical is θ, R_2 has coordinates

$(acos\theta - (b - a\theta)sin\theta, asin\theta + (b - a\theta)cos\theta)$

2nd Part

Writing $x = acos\theta - (b - a\theta)sin\theta$

and $y = asin\theta + (b - a\theta)cos\theta$,

$\dot{x} = -asin\theta.\dot{\theta} - [(-a\dot{\theta})sin\theta + (b - a\theta)cos\theta.\dot{\theta}]$

$= -(b - a\theta)cos\theta.\dot{\theta}$

and $\dot{y} = acos\theta.\dot{\theta} + [-a\dot{\theta}cos\theta + (b - a\theta)(-sin\theta).\dot{\theta}]$

$= -(b - a\theta)sin\theta.\dot{\theta}$

Then speed at R_2 is $\sqrt{\dot{x}^2 + \dot{y}^2} = (b - a\theta)\dot{\theta}$

(ii) The table is smooth, and the only external force on the system is the reaction force from the cylinder, which does no work, as there is no motion of the string at the cylinder. Hence the kinetic energy of the particle is conserved, and therefore its speed is constant.

So $(b - a\theta)\dot{\theta} = u$, as required.

[Alternatively, conservation of energy can be applied to just the particle, noting that the only force on the particle is the tension in the string, but that this force does no work, as the particle cannot move in the direction of the (straight part of the) string.]

(iii) First of all,

$(b - a\theta)\dot{\theta} = u \Rightarrow \frac{d\theta}{dt} = \frac{u}{b-a\theta}$

$\Rightarrow \int (b - a\theta)d\theta = u \int dt$

$\Rightarrow b\theta - \frac{1}{2}a\theta^2 = ut + C$

When $t = 0, \theta = 0$, so that $C = 0$,

and $2ut = \theta(2b - a\theta)$ (**)

Let T be the tension in the string.

Then, by N2L (resolving the acceleration along the string):

$T = m(\ddot{x}\sin\theta - \ddot{y}cos\theta)$, where $\ddot{x} = \frac{d}{dt}[-(b - a\theta)cos\theta.\dot{\theta}]$

$= -(-a\dot{\theta})cos\theta.\dot{\theta} - (b - a\theta)(-sin\theta.\dot{\theta})\,\dot{\theta} - (b - a\theta)cos\theta.\ddot{\theta}$

by Leibnitz's differentiation rule for products of more than two functions;

and $\ddot{y} = \frac{d}{dt}[-(b - a\theta)sin\theta.\dot{\theta}]$

$= -(-a\dot{\theta})sin\theta.\dot{\theta} - (b - a\theta)(cos\theta.\dot{\theta})\,\dot{\theta} - (b - a\theta)sin\theta.\ddot{\theta}$

Hence $\frac{T}{m} =$

$[-(-a\dot{\theta})cos\theta.\dot{\theta} - (b - a\theta)(-sin\theta.\dot{\theta})\,\dot{\theta} - (b - a\theta)cos\theta.\ddot{\theta}]sin\theta$

$-[a\dot{\theta}sin\theta.\dot{\theta} - (b - a\theta)(cos\theta.\dot{\theta})\,\dot{\theta} - (b - a\theta)sin\theta.\ddot{\theta}]cos\theta$

$= \dot{\theta}^2(asin\theta cos\theta + bsin^2\theta - a\theta sin^2\theta - asin\theta cos\theta + bcos^2\theta - a\theta cos^2\theta) + \ddot{\theta}(-bcos\theta sin\theta + a\theta cos\theta sin\theta + bsin\theta cos\theta - a\theta sin\theta cos\theta)$

$= \dot{\theta}^2(b - a\theta)$

And so, as $(b - a\theta)\dot{\theta} = u$,

$T = m(\frac{u}{b-a\theta})^2(b - a\theta) = \frac{mu^2}{b-a\theta}$ (***)

Then result to prove is $b - a\theta = \sqrt{b^2 - 2aut}$

or $(b - a\theta)^2 = b^2 - 2aut$;

ie $-2ab\theta + a^2\theta^2 = -2aut$

or $2ut = 2b\theta - a\theta^2 = \theta(2b - a\theta)$,

which was established at (**).

Alternative method

Because the particle is always moving perpendicular to the (straight part of the) string, it is instantaneously performing circular motion (ie the centre of the circle is changing).

The tension in the string is the only force causing the circular motion. So, if v is the speed of the particle,

$T = m\frac{v^2}{b-a\theta}$ (as the instantaneous radius is $R_1R_2 = b - a\theta$)

Then, as the speed is constant, $T = m\frac{u^2}{b-a\theta}$, which is (***), and we can then prove that $b - a\theta = \sqrt{b^2 - 2aut}$, as above.