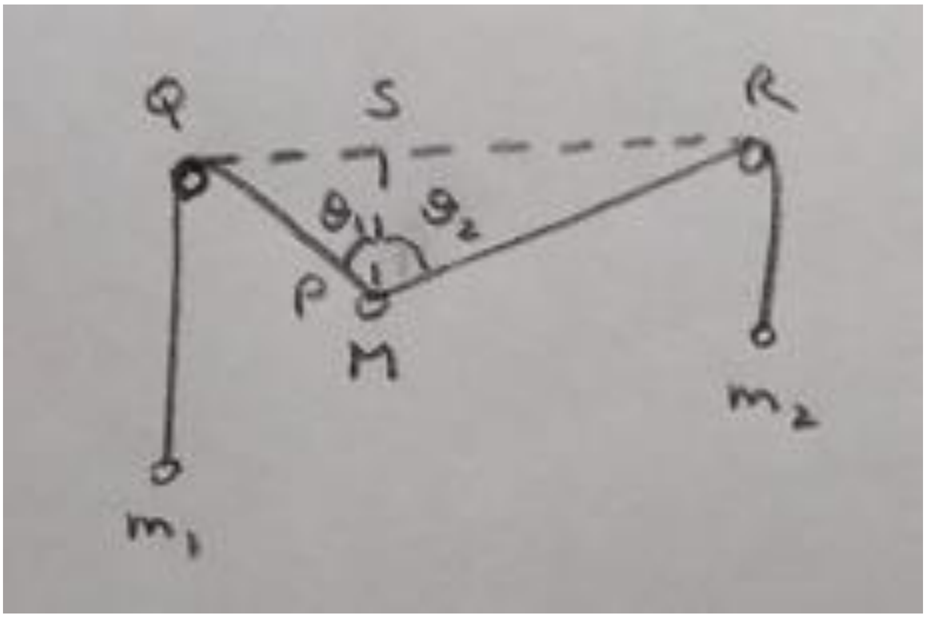
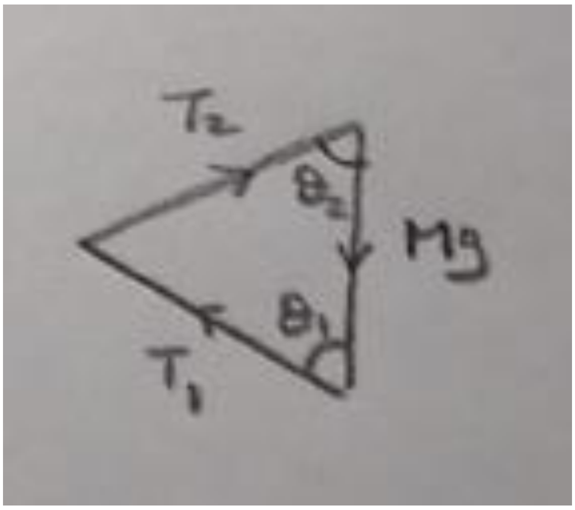
STEP 2021, P2, Q9 - Solution (3 pages; 18/2/23)

(i)(a)



[It isn't immediately clear whether S will be nearer Q or R. It has been provisionally placed closer to Q.]



As the particle of mass M is in equilibrium, the resultant of the force vectors on it is zero, and so a triangle of forces can be created, as shown.

From this, $|Mg| < |T_1| + |T_2|$,

Then, as $T_1 = m_1 g$ & $T_2 = m_2 g$ (from the equilibrium of the particles of masses m_1 and m_2), $M < m_1 + m_2$.

Also, as the angle between the sides $|Mg|$ and $|T_2|$ is θ_2,

$|T_1|^2 = |T_2|^2 + |Mg|^2 - 2|T_2||Mg|cos\theta_2$ (by the Cosine rule)

$< |T_2|^2 + |Mg|^2$, as θ_2 is acute;

and hence $m_1{}^2 < m_2{}^2 + M^2$, so that $m_1{}^2 - m_2{}^2 < M^2$

and $\sqrt{m_1{}^2 - m_2{}^2} < M$

Thus $\sqrt{m_1{}^2 - m_2{}^2} < M < m_1 + m_2$, as required.

(b) $\frac{QS}{SR} = \frac{SPtan\theta_1}{SPtan\theta_2} = \frac{tan\theta_1}{tan\theta_2}$ (*)

Once again, applying the Cosine rule to the vector triangle,

$|T_1|^2 = |T_2|^2 + |Mg|^2 - 2|T_2||Mg|cos\theta_2$

and also $|T_2|^2 = |T_1|^2 + |Mg|^2 - 2|T_1||Mg|cos\theta_1$

Hence $m_1{}^2 = m_2{}^2 + M^2 - 2m_2Mcos\theta_2$

and $m_2{}^2 = m_1{}^2 + M^2 - 2m_1Mcos\theta_1$

Then $r = \frac{m_2{}^2 + M^2 - m_1{}^2}{m_1{}^2 + M^2 - m_2{}^2} = \frac{2m_2Mcos\theta_2}{2m_1Mcos\theta_1} = \frac{m_2cos\theta_2}{m_1cos\theta_1}$ (**)

By N2L horizontally (applied to the particle of mass M) ,
$T_1sin\theta_1 = T_2sin\theta_2$, so that $\frac{sin\theta_1}{sin\theta_2} = \frac{m_2}{m_1}$ (***)

Then, from (*), $\frac{QS}{SR} = \frac{tan\theta_1}{tan\theta_2} = \frac{sin\theta_1}{sin\theta_2} . \frac{cos\theta_2}{cos\theta_1} = \frac{m_2}{m_1} . \frac{cos\theta_2}{cos\theta_1} = r$, from (**), as required.

(ii) **1st Part**

$M^2 = {m_1}^2 + {m_2}^2 \Rightarrow (Mg)^2 = {T_1}^2 + {T_2}^2$

Then, from the triangle of forces, the angle between T_1 & T_2 is 90°, by Pythagoras. Hence $\theta_1 + \theta_2 = 90°$, as required.

2nd Part

$\frac{QR}{SP} = \frac{QS}{SP} + \frac{SR}{SP} = tan\theta_1 + tan\theta_2$

$= tan\theta_1 + tan(90 - \theta_1)$

$= tan\theta_1 + \frac{1}{tan\theta_1}$

Also, $\frac{sin\theta_1}{sin\theta_2} = \frac{m_2}{m_1}$, from (***)

so that $\frac{m_2}{m_1} = \frac{sin\theta_1}{sin(90-\theta_1)} = \frac{sin\theta_1}{cos\theta_1} = tan\theta_1$

Hence $\frac{QR}{SP} = \frac{m_2}{m_1} + \frac{m_1}{m_2} = \frac{{m_2}^2 + {m_1}^2}{m_1 m_2} = \frac{M^2}{m_1 m_2}$