STEP 2021, P2, Q2 - Solution (3 pages; 17/3/23)

(i) Result to prove: $3pq - p^3 - 2r = 0$

$LHS = 3(a+b)(a^2+b^2) - (a+b)^3 - 2(a^3+b^3)$

[Note: $a+b$ must be a factor of a^3+b^3: let $f(x) = x^3 + b^3$; then $f(-b) = 0$, so that $x+b$ is a factor of $f(x)$, and hence $a+b$ is a factor of $f(a)$.]

$= (a+b)\{3(a^2+b^2) - (a+b)^2 - 2(a^2 - ab + b^2)\}$

$= (a+b)\{0\} = 0$, as required.

(ii) If the roots of $2x^2 - 2px + (p^2 - q) = 0$ (*) are a & b, then

$a + b = -\left(\frac{-2p}{2}\right) = p,$

and $ab = \frac{p^2-q}{2}$, so that $a^2 + b^2 = (a+b)^2 - 2ab$

$= p^2 - (p^2 - q) = q$

And $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$= p([a^2+b^2] - ab) = p\left(q - \frac{p^2-q}{2}\right) = \frac{p}{2}(3q - p^2)$

$= \frac{1}{2}(3pq - p^3) = \frac{1}{2}(2r) = r$

So, a & b can be chosen to be the roots of (*).

(iii) **1st Part**

[We have a dilemma here: do we (a) try to use (i) straightaway, and hope that it leads somewhere, or (b) start using the standard relations connecting the roots and coefficients of polynomials, and hope that (i) will be needed at some point. With hindsight, it turns out that (b) leads to the required result, but *without* using (i) [see below], and so (a) is in fact the best approach!]

The given conditions can be rewritten as

$a + b = s - c$, $a^2 + b^2 = t - c^2$ & $a^3 + b^3 = u - c^3$

Then, writing $p = s - c, q = t - c^2$ & $r = u - c^3$, and applying the result in (i),

$3pq - p^3 = 2r \Rightarrow 3(s - c)(t - c^2) - (s - c)^3 = 2(u - c^3)$

$\Rightarrow (3st - 3sc^2 - 3ct + 3c^3) - (s^3 - 3s^2c + 3sc^2 - c^3)$

$-2u + 2c^3 = 0$

or $6c^3 - 6sc^2 + (-3t + 3s^2)c + 3st - s^3 - 2u = 0$,

and so c is a root of the required eq'n.

2nd Part

By symmetry, the other 2 roots are a & b.

3rd Part

As the roots satisfy $abc = -\frac{3st - s^3 - 2u}{6}$, and as $abc = v$, it follows that $6v = s^3 - 3st + 2u$, as required.

[It is also possible to show that a, b & c are roots by verifying that

$a + b + c = -\frac{(-6s)}{s}$, as well as the other standard relations connecting the roots and coefficients of polynomials. The Examiner's Report says that "such attempts could not score credit for showing c was a root", with the implication that the proof isn't rigorous enough. In reality, it probably *is* rigorous enough for

STEP purposes, but the marks are being withheld because the specified method isn't being used.]

(iv) **1st Part**

From (iii), a, b & c will satisfy the cubic

$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$,

with $s = 3, t = 1, u = -3$ & $v = 2$;

so that $6x^3 - 18x^2 + 24x - 12 = 0$,

or $x^3 - 3x^2 + 4x - 2 = 0$

By inspection, $x = 1$ is a root, giving

$(x - 1)(x^2 + kx + 2) = 0$,

so that, equating coefficients of x, $2 - k = 4$, so that $k = -2$,

giving $(x - 1)(x^2 - 2x + 2) = 0$,

so that the other 2 roots are $\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

and we can set $a = 1, b = 1 + i$ & $c = 1 - i$

2nd Part

Verifying that these satisfy the given conditions:

$a + b + c = 1 + 1 + i + 1 - i = 3$

$a^2 + b^2 + c^2 = 1 + (1 - 1 + 2i) + (1 - 1 - 2i) = 1$

$a^3 + b^3 + c^3 = 1 + 2i(1 + i) - 2i(1 - i) = 1 + 2i - 2 - 2i - 2$

$= -3$

and $abc = 1(1 + i)(1 - i) = 2$

So a, b & c satisfy (**).