STEP 2021, P2, Q12 - Solution (4 pages; 1/3/23)

(i) $P(A\ wins) = \sum_{k=0}^{\infty} P(k\ consecutive\ draws) \times p_A$

$= \sum_{k=0}^{\infty}(1 - p_A - p_B)^k p_A$

$= \frac{p_A}{1-[1-p_A-p_B]} = \frac{p_A}{p_A+p_B}$ (sum of infinite Geometric series),

as required.

(ii) **Part 1**

Consider the circumstances necessary for the match to continue:

If the 1st game is won by A, then the 2nd game must be won by B (otherwise A wins). Similarly, if the 1st game is won by B, then the 2nd game must be won by A (otherwise B wins). After 2 games the starting position is the same, as each player has won one game.

A match therefore consists of a sequence of one or more pairs AB or BA, followed by either AA or BB. So there will be an even number of games.

Part 2

$P(a\ pair\ AB\ or\ BA) = pq + qp = 2pq$

and so $P(A\ wins) = \sum_{k=0}^{\infty}(2pq)^k p^2 = \frac{p^2}{1-2pq} = \frac{p^2}{(p+q)^2-2pq}$

$= \frac{p^2}{p^2+q^2}$, as required.

[The Official Sol'n uses (i) and considers pairs of games, so that AA represents a win for A, BB represents a loss, and AB/BA represents a draw. Then $p_A = p^2, p_B = q^2$, and

$P(A\ wins) = \frac{p^2}{p^2+q^2}$]

(iii) **Part 1** (Cautious version)

The player has to win the next round, otherwise they will have no tokens and lose; so

$P(player\ wins) = P(player\ wins\ next\ round) \times$

$P(player\ wins|they\ start\ with\ a\ surplus\ of\ 2)$ (*)

Starting at position AA, we can consider the possible outcomes for the next pair of events:

AA (A wins)

BB (A loses, as they now have no tokens)

AB or BA (A still has a surplus of 2; ie 'a draw')

So the situation is the same as in (i) with $p_A = p^2, p_B = q^2$

and so $P(player\ wins|they\ start\ with\ a\ surplus\ of\ 2) = \frac{p^2}{p^2+q^2}$

Hence, from (*), $P(player\ wins) = p.\frac{p^2}{p^2+q^2} = \frac{p^3}{p^2+q^2}$

Part 2 (Bold version)

$P(player\ wins) = P(AA).P(AAAA|AA) = p^2$

Part 3 (comparison of versions)

[The question is slightly ambiguous: it could possibly mean "show that the player is more likely to win **than not** in the cautious version when $1 > p > \frac{1}{2}$; but this would imply that they are more likely to win **than not** in the bold version when $0 < p < \frac{1}{2}$, but clearly the probability of winning increases with p, so we can reject this interpretation.]

[Note that the question is asking us to show that if $\frac{1}{2} < p < 1$ (X, say), then the probability of winning is greater for the cautious version than for the bold version (Y, say), so we must prove that

$X \Rightarrow Y$. However, this seems to be difficult to do directly. Instead we can show that $X \Leftrightarrow Y$ (ie X and Y are equivalent), and then deduce that $X \Rightarrow Y$, as long as there is convincing equivalence at each step of the argument (and we word things carefully).]

The probability of winning is greater for the cautious version than for the bold version when

$\frac{p^3}{p^2+q^2} > p^2$

$\Leftrightarrow p > p^2 + q^2 = (p+q)^2 - 2pq = 1 - 2pq$ (assuming $p \neq 0$)

$\Leftrightarrow p + 2pq > 1$

$\Leftrightarrow p + 2p(1-p) > 1,$

$\Leftrightarrow 2p^2 - 3p + 1 < 0$

$\Leftrightarrow (2p-1)(p-1) < 0$

$\Leftrightarrow 2p - 1 > 0$ & $p - 1 < 0$, as $p - 1 \leq 0$

$\Leftrightarrow p > \frac{1}{2}$ and $p < 1$; ie $\frac{1}{2} < p < 1$

Hence $\frac{1}{2} < p < 1 \Rightarrow$ the probability of winning is greater for the cautious version than for the bold version.

And the probability of winning is greater for the bold version than for the cautious version when

$(2p - 1)(p - 1) > 0$

$\Leftrightarrow 2p - 1 < 0$ & $p - 1 < 0$, as $p - 1 \leq 0$

$\Leftrightarrow p < \frac{1}{2}$

In the case of $p = 0$ (excluded earlier), the two versions both have the same probability (of zero), and so the required condition here becomes $0 < p < \frac{1}{2}$.

Hence $0 < p < \frac{1}{2} \Rightarrow$ the probability of winning is greater for the bold version than for the cautious version.