STEP 2021, P2, Q11 - Solution (4 pages; 22/2/23)

(i) **1st Part**

$P_2 = \frac{1}{2}$ (the probability that T_1 sits in S_1)

2nd Part

$P_3 = P(T_1 \text{ sits in } S_1) \times 1$

$+P(T_1 \text{ sits in } S_2) \times P(T_2 \text{ sits in } S_1 | T_1 \text{ sits in } S_2)$

$+P(T_1 \text{ sits in } S_3) \times 0$

$= \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$

(ii) **1st Part**

If T_1 sits in S_k (where $k \leq n-1$) then $T_2, \dots, T_{k-1}$ will sit in their allocated seats (for $k \geq 3$). T_k then has to choose their seat at random, from seats

$1, k+1, k+2, \dots n$ (a total of $n-(k-1) = n-k+1$ seats),

and the situation is then the same as if T_k is the 1st passenger arriving, and there are $n-k+1$ passengers in total;

so that $P(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) = P_{n-k+1}$

If T_1 sits in S_2, then T_2 has to choose their seat at random, from seats $1, 3, 4, 5, \dots n$; a total of $n-1$ seats. And, when $k=2$,

$n-k+1 = n-1$.

Thus, $P(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) = P_{n-k+1}$ for $2 \leq k \leq n-1$ (with $n \geq 3$, so that $n-1 \geq 2$), as required.

2nd Part

$P_n = \sum_{k=1}^{n} P(T_1 \text{ sits in } S_k).P(T_n \text{ sits in } S_n|T_1 \text{ sits in } S_k)$

$= P(T_1 \text{ sits in } S_1).P(T_n \text{ sits in } S_n|T_1 \text{ sits in } S_1)$

$+ \sum_{k=2}^{n-1} P(T_1 \text{ sits in } S_k).P(T_n \text{ sits in } S_n|T_1 \text{ sits in } S_k)$

$+P(T_1 \text{ sits in } S_n).P(T_n \text{ sits in } S_n|T_1 \text{ sits in } S_n)$

[Note that the result proved in the 1st Part only applies for

$2 \leq k \leq n-1$]

$= \frac{1}{n}.1 + (\sum_{k=2}^{n-1} \frac{1}{n}.P_{n-k+1}) + \frac{1}{n}.0$

Then, writing $r = n - k + 1$,

$P_n = \frac{1}{n} + \frac{1}{n}\sum_{r=n-1}^{2} P_r$

$= \frac{1}{n}(1 + \sum_{r=2}^{n-1} P_r)$, as required (for $n \geq 3$)

(iii) **1st Part**

$P_4 = \frac{1}{4}(1 + P_2 + P_3) = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$

and $P_5 = \frac{1}{5}(1 + P_2 + P_3 + P_4) = \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$

We can conjecture that $P_n = \frac{1}{2}$

2nd Part

Assume that $P_k = \frac{1}{2}$ (for $k \geq 2$)

Then, from the 2nd Part of (ii), with $k + 1 \geq 3$,

$P_{k+1} = \frac{1}{k+1}\left(1 + ([k+1] - 2)\left(\frac{1}{2}\right)\right)$

$= \frac{1}{2(k+1)}(2+k-1) = \frac{1}{2}$

So, if $P_k = \frac{1}{2}$ (with $k \geq 2$) , then $P_{k+1} = \frac{1}{2}$

As $P_2 = \frac{1}{2}$, it follows that $P_3 = \frac{1}{2}$, $P_4 = \frac{1}{2}$, … ,

and so, by the principle of induction, $P_n = \frac{1}{2}$ for all $n \geq 2$

(iv) [In the same way as for the 1st Part of (ii),]

$P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_k) = Q_{n-k+1}$,

provided now that $2 \leq k \leq n-2$ (with $n \geq 4$, so that $n-2 \geq 2$),

Then $Q_n = \sum_{k=1}^{n} P(T_1$ sits in $S_k).P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_k)$

$= P(T_1$ sits in $S_1).P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_1)$

$+ \sum_{k=2}^{n-2} P(T_1$ sits in $S_k).P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_k)$

$+P(T_1$ sits in $S_{n-1}).P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_{n-1})$

$+P(T_1$ sits in $S_n).P(T_{n-1}$ sits in $S_{n-1}|T_1$ sits in $S_n)$

$= \frac{1}{n} .1 + (\sum_{k=2}^{n-2} \frac{1}{n}.Q_{n-k+1}) + \frac{1}{n}.0 + \frac{1}{n}.1$

Then, writing $r = n-k+1$,

$Q_n = \frac{1}{n}(2 + \sum_{r=n-1}^{3} Q_r)$

$= \frac{1}{n}(2 + \sum_{r=3}^{n-1} Q_r)$ (for $n \geq 4$)

Now, $Q_2 = P(T_1$ sits in $S_1) = \frac{1}{2}$

and $Q_3 = P(T_1$ sits in $S_1) \times 1$

$+P(T_1$ sits in $S_2) \times 0$

$+P(T_1 \text{ sits in } S_3) \times 1$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

So $Q_4 = \frac{1}{4}(2 + Q_3) = \frac{1}{4}\left(2 + \frac{2}{3}\right) = \frac{2}{3}$

and $Q_5 = \frac{1}{5}(2 + Q_3 + Q_4) = \frac{1}{5}\left(2 + \frac{4}{3}\right) = \frac{2}{3}$

To prove by induction that $Q_n = \frac{2}{3}$, for $n \geq 4$:

Assume that $Q_k = \frac{2}{3}$.

Then $Q_{k+1} = \frac{1}{k+1}(2 + Q_3 + Q_4 + \cdots + Q_k)$

$= \frac{1}{k+1}(2 + (k-2)\left(\frac{2}{3}\right))$

$= \frac{2}{3(k+1)}(3 + k - 2) = \frac{2}{3}$

So, if $Q_k = \frac{2}{3}$, then $Q_{k+1} = \frac{2}{3}$.

As $Q_4 = \frac{2}{3}$, it follows by the principle of induction that

$Q_n = \frac{2}{3}$ for $n \geq 4$

Also (as already established), $Q_2 = \frac{1}{2}$ and $Q_3 = \frac{2}{3}$