STEP 2021, P2, Q10 - Solution (4 pages; 7/3/23)

(i) At the Origin, the only forces on the bead (relative to the ground) are gravity and the normal reaction from the wire, so

that there is no horizontal acceleration relative to the ground. So, as the train is itself accelerating horizontally relative to the ground, the bead will not be stationary relative to the train.

(ii) **1st Part – Approach 1**

[If we suspect that differentiation is involved, we could consider $I = \int \dot{x}\ddot{x}dt = \dot{x}\,\dot{x} - \int \ddot{x}\dot{x}dt$, so that $2I = \dot{x}^2$, which could prompt consideration of the kinetic energy component $\frac{1}{2}m\dot{x}^2$]

Relative to the frame of reference of the wire, the bead has a ('fictitious') force F and acceleration a applied to it (both directed East); where $F = ma$, if m is the mass of the bead.

Then the work done on the bead equals its gain in kinetic energy and potential energy:

$Fx = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$,

so that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy - ax = 0$

Then differentiating wrt t:

$\dot{x}\ddot{x} + \dot{y}\ddot{y} + g\dot{y} - a\dot{x} = 0$

or $\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$, as required. (*)

1st Part – Approach 2

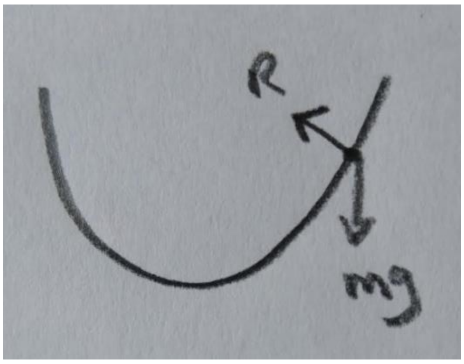
Let (X, y) be the coordinates of the bead relative to the ground.

(x, y) are the coordinates of the bead relative to the wire (on the train),

and, at time t, $X = X_0{}^T - V_0{}^T t - \frac{1}{2}at^2 + x$ (1),

where $X_0{}^T$ and $V_0{}^T$ are the initial displacement and speed of the train (relative to the ground), noting that the train is travelling in the negative X direction

We can create a force diagram for the bead, relative to the ground (so that no 'fictitious' forces need be considered).



The forces that act on the bead consist of gravity, and the normal reaction force from the wire. (There is no frictional component of the reaction force, as the wire is smooth.)

Because the bead is constrained to move on the wire (and the wire is fixed in the train), its motion will be determined solely by the components of the forces along the wire.

The normal reaction force has no component along the wire, and the component of gravity along the wire is $mgsin\theta$, where $tan\theta$ is the gradient of the tangent to the curve.

Applying N2L in the X and y directions:

$-mgsin\theta.cos\theta = m\ddot{X}$ (2)

and $-mgsin\theta.sin\theta = m\ddot{y}$ (3)

Result to prove: $\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$

Equivalently, $\frac{\dot{y}}{\dot{x}} = \frac{a-\ddot{x}}{\ddot{y}+g}$ (4)

Now $\frac{\dot{y}}{\dot{x}} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{dy}{dx} = tan\theta$, and so (4) can be written as

$\frac{a-\ddot{x}}{\ddot{y}+g} = tan\theta$ (5)

Now, differentiating (1) gives $\dot{X} = -V_0{}^T - at + \dot{x}$,

and differentiating again gives $\ddot{X} = -a + \ddot{x}$

Then, from (2) & (3),

$$\frac{a-\ddot{x}}{\ddot{y}+g} = \frac{-\ddot{X}}{\ddot{y}+g} = \frac{gsin\theta.cos\theta}{-gsin\theta.sin\theta+g} = \frac{sin\theta.cos\theta}{1-sin^2\theta} = \frac{sin\theta.cos\theta}{cos^2\theta} = tan\theta,$$

as required in (5)

2nd Part

$\frac{d}{dt}\left\{\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy\right\} = \dot{x}\ddot{x} + \dot{y}\ddot{y} - a\dot{x} + g\dot{y} = 0$, from (*),

and so $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant, as required.

(iii) When the bead reaches its maximum vertical displacement, its speed relative to the wire will be zero; ie $\dot{x} = \dot{y} = 0$

Then $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy = c$ (where c is a constant) (**)

becomes $-ax + g.\frac{x^2}{k} = c$

Also, $x = 0$ initially (when the speed is also zero), and so $c = 0$.

Then $-ax + g.\frac{x^2}{k} = 0$, and (with $x \neq 0$) $-a + \frac{gx}{k} = 0$,

so that $x = \frac{ak}{g}$, and hence $b = \frac{(\frac{ak}{g})^2}{k} = \frac{a^2k}{g^2}$

(iv) **1st Part**

The speed of the bead is greatest when

$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) = ax - gy$ is greatest (from (**))

ie when $ax - g.\frac{x^2}{k}$ is greatest.

The roots of $ax - g.\frac{x^2}{k} = 0$ occur at $x = 0$ & $x = \frac{ak}{g}$,

and so the maximum of $ax - g.\frac{x^2}{k}$ occurs midway between these roots, at $\frac{ak}{2g}$

2nd Part

From (**), $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy = 0$, so that the speed of the bead is $\sqrt{2(ax - gy)}$

When $x = \frac{ak}{2g}$, $y = \frac{1}{k}x^2 = \frac{a^2k}{4g^2}$, and so the maximum speed is

$\sqrt{2(ax - gy)} = \sqrt{\frac{a^2k}{g} - \frac{a^2k}{2g}} = \sqrt{\frac{a^2k}{2g}}$ or $a\sqrt{\frac{k}{2g}}$