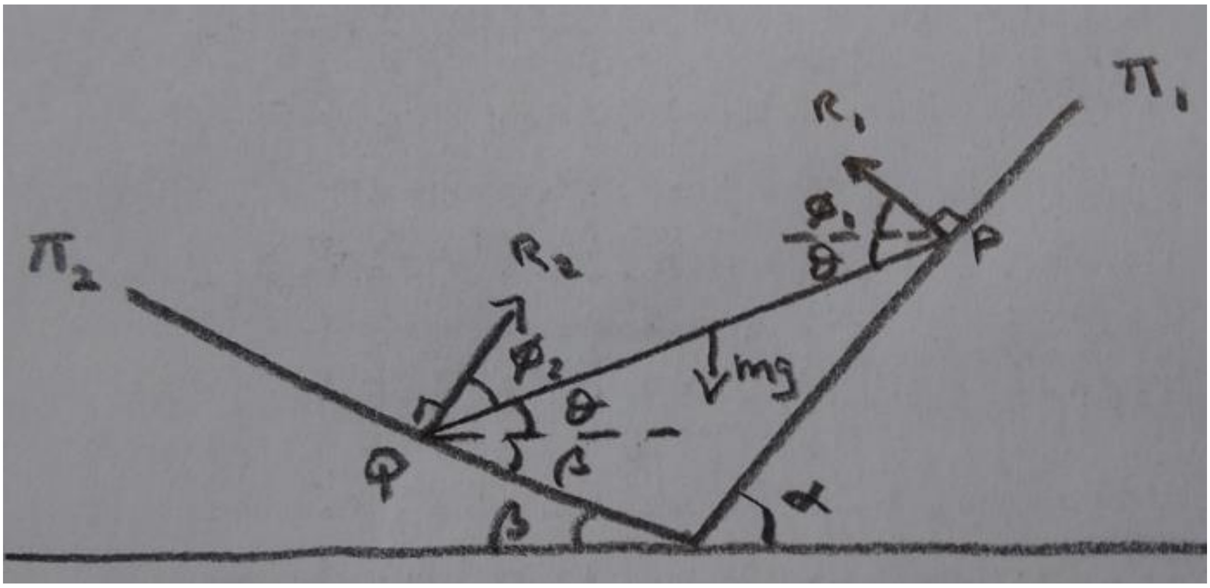
STEP 2020, P3, Q9 - Solution (5 pages; 15/1/23)

(i)



A force diagram has been drawn for the rod. If the planes are smooth, the only forces on the rod are the normal reactions R_1 at P and R_2 at Q, as well as the weight of the rod, mg.

[In general terms, 3 equations can be obtained, of the following forms:

(i) Resolving horizontally: $a(\theta)R_1 + b(\theta)R_2 = 0$

(ii) Resolving vertically: $c(\theta)R_1 + d(\theta)R_2 = mg$

(iii) Taking moments about the midpoint of the rod:

$e(\theta)R_1 + f(\theta)R_2 = 0$

We see that any solution must be of the form:

$R_1 = h(\theta)mg, R_2 = i(\theta)mg$

Also, referring to equations (i) & (iii): in order for there to be a solution where R_1 and R_2 are not both zero, we require $\begin{vmatrix} a(\theta) & b(\theta) \\ e(\theta) & f(\theta) \end{vmatrix} = 0$ [otherwise there would just be the single solution $R_1 = R_2 = 0$]

[Alternatively, $\frac{R_1}{R_2} = -\frac{b(\theta)}{a(\theta)}$ from (i), and $\frac{R_1}{R_2} = -\frac{f(\theta)}{e(\theta)}$ from (iii), so that $-\frac{b(\theta)}{a(\theta)} = -\frac{f(\theta)}{e(\theta)}$, and hence $a(\theta)f(\theta) - b(\theta)e(\theta) = 0$]

and this places a constraint on θ.]

The extra angles ϕ_1 & ϕ_2 have been introduced.

$\phi_1 + \theta + (\pi - [\theta + \beta] - [\pi - \alpha - \beta]) = \frac{\pi}{2}$,

so that $\phi_1 = \frac{\pi}{2} - \alpha$

And $\phi_2 + \theta + \beta = \frac{\pi}{2}$, so that $\phi_2 = \frac{\pi}{2} - \theta - \beta$

Resolving forces horizontally,

$R_1 cos\phi_1 = R_2 \cos(\phi_2 + \theta)$

so that $\frac{R_1}{R_2} = \frac{\cos(\frac{\pi}{2} - \beta)}{\cos(\frac{\pi}{2} - \alpha)} = \frac{sin\beta}{sin\alpha}$

Taking moments about the midpoint of the rod (of length l, say):

$R_1 \sin(\theta + \phi_1) = R_2 sin\phi_2$,

so that $\frac{R_1}{R_2} = \frac{\sin(\frac{\pi}{2} - \theta - \beta)}{\sin(\theta + \frac{\pi}{2} - \alpha)} = \frac{\cos(\theta + \beta)}{\cos(\alpha - \theta)}$

Equating the two expressions for $\frac{R_1}{R_2}$: $\frac{sin\beta}{sin\alpha} = \frac{\cos(\theta+\beta)}{\cos(\alpha-\theta)}$

$\Rightarrow sin\beta(cos\alpha cos\theta + sin\alpha sin\theta) = sin\alpha(cos\theta cos\beta - sin\theta sin\beta)$

dividing by $cos\beta cos\alpha cos\theta$ [aiming to write in terms of $tan\theta, tan\alpha$ & $tan\beta$ initially]:

$\Rightarrow tan\beta(1 + tan\alpha tan\theta) = tan\alpha(1 - tan\theta tan\beta)$

[then, aiming to make $tan\theta$ the subject:]

$\Rightarrow tan\theta(2tan\alpha tan\beta) = tan\alpha - tan\beta$

$\Rightarrow 2tan\theta = cot\beta - cot\alpha$, as required.

(ii) Now there is an additional force, μR_2 acting down the plane Π_2 (the rod is on the point of slipping down Π_1; ie slipping up Π_1, and friction opposes motion or attempted motion), and in the diagram θ is replaced by ϕ.

As before, $\phi_1 = \frac{\pi}{2} - \alpha$, but now $\phi_2 = \frac{\pi}{2} - \phi - \beta$

Resolving forces horizontally,

$$R_1 cos\phi_1 = R_2 \cos(\phi_2 + \phi) + \mu R_2 \cos\left(\frac{\pi}{2} - [\phi + \phi_2]\right)$$

so that $\frac{R_1}{R_2} = \frac{\cos\left(\frac{\pi}{2} - \beta\right) + \mu\cos\beta}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{sin\beta - \mu\cos\beta}{sin\alpha}$

Taking moments about the midpoint of the rod:

$$R_1 \sin(\phi + \phi_1) = R_2 sin\phi_2 - \mu R_2 cos\phi_2 \ ,$$

so that $\frac{R_1}{R_2} = \frac{\sin\left(\frac{\pi}{2} - \phi - \beta\right) - \mu\cos\left(\frac{\pi}{2} - \phi - \beta\right)}{\sin\left(\phi + \frac{\pi}{2} - \alpha\right)} = \frac{\cos(\phi + \beta) - \mu\sin(\phi + \beta)}{\cos(\alpha - \phi)}$

Equating the two expressions for $\frac{R_1}{R_2}$:

$$\frac{sin\beta + \mu\cos\beta}{sin\alpha} = \frac{\cos(\phi + \beta) - \mu\sin(\phi + \beta)}{\cos(\alpha - \phi)}$$

$\Rightarrow (sin\beta + \mu\cos\beta)(cos\alpha cos\phi + sin\alpha sin\phi) =$

$[(cos\phi cos\beta - sin\phi sin\beta) - \mu(sin\phi cos\beta + cos\phi sin\beta)]sin\alpha$

$\Rightarrow (tan\beta + \mu)(1 + tan\alpha tan\phi) =$

$[(1 - tan\phi tan\beta) - \mu(tan\phi + tan\beta)]tan\alpha$

$\Rightarrow tan\beta + tan\beta tan\alpha tan\phi + \mu + \mu tan\alpha tan\phi$

$= tan\alpha - tan\alpha tan\phi tan\beta - \mu tan\phi tan\alpha - \mu tan\beta tan\alpha$

$\Rightarrow tan\beta + 2tan\beta tan\alpha tan\phi + \mu + 2\mu tan\alpha tan\phi$

$= tan\alpha - \mu tan\beta tan\alpha$

[then, aiming to make $tan\phi$ the subject, as the result to be proved is: $tan\phi = \frac{1}{2}(cot\beta - cot\alpha) - \frac{\mu}{(\mu+tan\beta)sin2\beta}$ (using the fact that

$2tan\theta = cot\beta - cot\alpha$)]

$\Rightarrow tan\phi\{2tan\beta tan\alpha + 2\mu tan\alpha)$

$= -tan\beta - \mu + tan\alpha - \mu tan\beta tan\alpha$

$\Rightarrow tan\phi = \frac{tan\alpha - tan\beta - \mu - \mu tan\alpha tan\beta}{2tan\alpha(tan\beta+\mu)}$

$= \frac{-(tan\beta+\mu)+tan\alpha(1-\mu tan\beta)}{2tan\alpha(tan\beta+\mu)}$

$= -\frac{1}{2}cot\alpha + \frac{(1-\mu tan\beta)}{2(tan\beta+\mu)}$

$= -\frac{1}{2}cot\alpha + (\frac{1}{2}cot\beta - \frac{1}{2tan\beta}) + \frac{(1-\mu tan\beta)}{2(tan\beta+\mu)}$

$= \frac{1}{2}(cot\beta - cot\alpha) + \frac{-(tan\beta+\mu)+tan\beta(1-\mu tan\beta)}{2tan\beta(tan\beta+\mu)}$

$= \frac{1}{2}(cot\beta - cot\alpha) + \frac{-\mu-\mu tan^2\beta}{2tan\beta(tan\beta+\mu)}$

$= \frac{1}{2}(cot\beta - cot\alpha) + \frac{\mu sec^2\beta}{2tan\beta(tan\beta+\mu)}$

$= \frac{1}{2}(cot\beta - cot\alpha) + \frac{\mu}{2sin\beta cos\beta(\mu+tan\beta)}$

$= \frac{1}{2}(cot\beta - cot\alpha) + \frac{\mu}{(\mu+tan\beta)sin2\beta}$, as required.