STEP 2020, P3, Q7 - Solution (3 pages; 11/8/22)

(i) Differentiating $\frac{dy}{dx} + g(x)y = u$ wrt x gives

$\frac{d^2y}{dx^2} + g'(x)y + g(x)\frac{dy}{dx} = \frac{du}{dx} = h(x) - f(x)u$

$= h(x) - f(x)\{\frac{dy}{dx} + g(x)y\}$,

and hence $\frac{d^2y}{dx^2} + \frac{dy}{dx}\{g(x) + f(x)\} + y\{g'(x) + f(x)g(x)\} = h(x)$,

which is the same as (1).

(ii) **1st Part**

Comparing (1) & (2), we require the following:

$g(x) + f(x) = 1 + \frac{4}{x}$ (A)

$g'(x) + f(x)g(x) = \frac{2}{x} + \frac{2}{x^2}$ (B)

Using (A) to eliminate $f(x)$ from (B):

$g'(x) + \{1 + \frac{4}{x} - g(x)\}g(x) = \frac{2}{x} + \frac{2}{x^2}$, (*)

which is the required 1st order equation.

2nd Part

Writing $g(x) = kx^n$, (*) becomes:

$knx^{n-1} + \left\{1 + \frac{4}{x} - kx^n\right\}kx^n = \frac{2}{x} + \frac{2}{x^2}$

Multiplying by x^2:

$knx^{n+1} + kx^{n+2} + 4kx^{n+1} - k^2x^{2n+2} = 2x + 2$

or $-k^2x^{2n+2} + kx^{n+2} + k(n+4)x^{n+1} = 2x + 2$

As this result must hold for all x,

let $n + 1 = 0$, in order to create a constant term on the LHS,

giving $-k^2 + kx + 3k = 2x + 2$,

so that $k = 2$, $n = -1$, and $g(x) = 2x^{-1}$

3rd Part

[The question (and official solution) seems to suggest that the pair of linear differential equations in (i) will be satisfied if the 2nd order differential equation in (i) is satisfied, but only the reverse implication was actually proved.]

If $g(x) = 2x^{-1}$, then as $g(x) + f(x) = 1 + \frac{4}{x}$, $f(x) = 1 + \frac{2}{x}$

Also, $h(x) = 4x + 12$

If we can assume that the pair of linear differential equations in (i) is satisfied, then

$\frac{du}{dx} + \left(1 + \frac{2}{x}\right)u = 4x + 12$, and $\frac{dy}{dx} + \frac{2}{x} \, .y = u$

The integrating factor for the 1st equation is $\exp\left(\int 1 + \frac{2}{x} dx\right)$

$= \exp(x + 2lnx) = x^2e^x$

Applying this to the 1st equation gives

$\frac{d}{dx}\,(ux^2e^x) = x^2e^x(4x + 12)$

$\Rightarrow ux^2e^x = \int e^x(4x^3 + 12x^2)dx$

This result of this integral will be of the form

$e^x(Ax^4 + Bx^3)$ (when we consider differentiating this function), and carrying out the differentiation gives:

$e^x([Ax^4 + Bx^3] + [4Ax^3 + 3Bx^2]) = e^x(4x^3 + 12x^2)$,

so that $A = 0, B = 4$

[This only works fortuitously; ie because the question has been designed this way.]

Thus $ux^2e^x = 4x^3e^x + C$

and $u = 4x + Cx^{-2}e^{-x}$

From the 2nd eq'n, $\frac{dy}{dx} + \frac{2}{x} \, . y = u$, and the given conditions,

$-3 + 2(5) = 4 + \frac{C}{e}$, so that $C = 3e$

And then from the 2nd eq'n again,

$\frac{dy}{dx} + \frac{2}{x} \, . y = 4x + 3x^{-2}e^{1-x}$

Multiplying both sides by x^2,

$\frac{d}{dx}\,(yx^2) = 4x^3 + 3e^{1-x}$,

so that $yx^2 = x^4 - 3e^{1-x} + D$

From the given conditions, $5 = 1 - 3 + D$, so that $D = 7$,

and so $y = x^2 - 3x^{-2}e^{1-x} + 7x^{-2}$

[The Examiner's Report refers to a 'complimentary' solution – they meant to say 'complementary' of course.]