STEP 2020, P3, Q2 - Solution (5 pages; 11/3/23)

(i) 1st Part

$sinhx + sinhy = 2k$

Differentiating both sides wrt x:

$coshx + coshy.\frac{dy}{dx} = 0$ (1)

$\Rightarrow \frac{dy}{dx} = -\frac{coshx}{coshy}$ (as $coshy \neq 0$) (2)

As $coshx \neq 0, \frac{dy}{dx} \neq 0$; ie C has no stationary points.

2nd Part

Differentiating both sides of (1) wrt x:

$sinhx + \left(sinhy.\frac{dy}{dx}\right)\frac{dy}{dx} + coshy\,.\frac{d^2y}{dx^2} = 0$ (3)

Then $\frac{d^2y}{dx^2} = 0 \Leftrightarrow sinhx + \left(sinhy.\frac{dy}{dx}\right)\frac{dy}{dx} = 0$ (as $coshy \neq 0$)

$\Leftrightarrow sinhx + sinhy(-\frac{coshx}{coshy})^2 = 0$, from (2)

$\Leftrightarrow sinhx(sinh^2y + 1) + sinhy(sinh^2x + 1) = 0$

$\Leftrightarrow sinhxsinhy(sinhy + sinhx) + (sinhx + sinhy) = 0$

$\Leftrightarrow (sinhx + sinhy)(sinhxsinhy + 1) = 0$

$\Leftrightarrow 2k(sinhxsinhy + 1) = 0$

$\Leftrightarrow 1 + sinhxsinhy = 0$ (as $k \neq 0$), as required.

3rd Part

A point of inflection is a turning point of the gradient. This means that $\frac{d^2y}{dx^2} = 0$ and if r is the smallest integer greater than 1 for which $\frac{d^ry}{dx^r} \neq 0$, then r is odd.

From the 2nd part, $\frac{d^2y}{dx^2} = 0 \Rightarrow 1 + sinhxsinhy = 0$

Then, as $sinhx + sinhy = 2k$,

$1 + sinhx(2k - sinhx) = 0$,

so that $sinh^2x - 2ksinhx - 1 = 0$,

and $sinhx = \frac{2k \pm \sqrt{4k^2+4}}{2} = k \pm \sqrt{k^2 + 1}$ (4)

To investigate $\frac{d^3y}{dx^3}$:

From (3), $sinhx + \left(sinhy.\frac{dy}{dx}\right)\frac{dy}{dx} + coshy\,.\frac{d^2y}{dx^2} = 0$

Differentiating both sides wrt x:

$coshx + coshy(\frac{dy}{dx})^2 + sinhy.2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + sinhy.\frac{d^2y}{dx^2}$

$+coshy.\frac{d^3y}{dx^3} = 0$

And, as $\frac{d^2y}{dx^2} = 0$ at a point of inflection,

$coshx + coshy(\frac{dy}{dx})^2 + coshy.\frac{d^3y}{dx^3} = 0$,

Suppose that $\frac{d^3y}{dx^3} = 0$.

Then $coshx + coshy\left(\frac{dy}{dx}\right)^2 = 0$, which is not possible, as $coshx, coshy$ and $\left(\frac{dy}{dx}\right)^2$ are all positive (as $\frac{dy}{dx} \neq 0$).

So $\frac{d^3y}{dx^3} \neq 0$ when $\frac{d^2y}{dx^2} = 0$, and (4) defines the points of inflection.

Hence $x = arsinh(k \pm \sqrt{k^2+1})$

$sinhx = \frac{2k \pm \sqrt{4k^2+4}}{2} = k \pm \sqrt{k^2+1}$

And, as $sinhx + sinhy = 2k$,

$y = arsinh\left(2k - \left[k \pm \sqrt{k^2+1}\right]\right) = arsinh(k \mp \sqrt{k^2+1})$

Thus the points of inflection are:

$(arsinh\left(k + \sqrt{k^2+1}\right), arsinh\left(k - \sqrt{k^2+1}\right))$

and $(arsinh\left(k - \sqrt{k^2+1}\right), arsinh\left(k + \sqrt{k^2+1}\right))$

(ii) **1st Part**

As $sinhx + sinhy = 2k$ and $x + y = a$,

$\frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^{a-x} - e^{x-a}) = 2k$

$\Rightarrow e^{2x} - 1 + e^a - e^{2x-a} = 4ke^x$

$\Rightarrow e^{2x}(1 - e^{-a}) - 4ke^x + (e^a - 1) = 0$, as required. (5)

2nd Part

First of all, $cosha \geq 1$ for any a.

Suppose that $a = 0$, so that $cosha = 1$.

Then (5) $\Rightarrow -4ke^x = 0$, which isn't possible. So $cosha > 1$.

Then, as (x, y)lies on both C and the line $x + y = a$, there must be a real sol'n of (5) for e^x, and so the discriminant of the quadratic must be non-negative;

ie $(-4k)^2 - 4(1-e^{-a})(e^a - 1) \geq 0$

$\Rightarrow 2k^2 - \frac{1}{2}(e^a - 1 - 1 + e^{-a}) \geq 0$

$\Rightarrow 2k^2 - cosha + 1 \geq 0$; ie $cosha \leq 2k^2 + 1$

Thus $1 < cosha \leq 2k^2 + 1$, as required.

(iii) To sketch C:

(1) The curve is symmetrical in x & y, and therefore about $y = x$

(2) From (i), there are points of inflection at A and B, where

$A = (arsinh(k + \sqrt{k^2+1}), arsinh(k - \sqrt{k^2+1}))$

and $B = (arsinh(k - \sqrt{k^2+1}), arsinh(k + \sqrt{k^2+1}))$;

noting that B is reflection of A in $y = x$;

also $k - \sqrt{k^2+1} < 0$, and $|k + \sqrt{k^2+1}| > |k - \sqrt{k^2+1}|$ (consider the positions of $k + \sqrt{k^2+1}$ and $k - \sqrt{k^2+1}$ on the number line, with $k > 0$), so that

$|arsinh(k + \sqrt{k^2+1})| > |arsinh(k - \sqrt{k^2+1})|$

(3) From (ii), the curve lies between $x + y = arcosh(1) = 0$ and

$x + y = arcosh(2k^2 + 1)$, with the curve touching

$x + y = arcosh(2k^2 + 1)$, but not touching $x + y = 0$

(4) $As\ x \to \infty, y \to \infty$, and as the curve doesn't touch $x + y = 0$, this must be the asymptote.

(5) By symmetry, the curve will meet $x + y = arcosh(2k^2 + 1)$ on the line $y = x$, when $x = \frac{1}{2}arcosh(2k^2 + 1)$;

ie C is $(\frac{1}{2}arcosh(2k^2+1), \frac{1}{2}arcosh(2k^2+1))$

(6) The curve meets the x-axis when $sinhx = 2k$; ie at

$D = (arsinh(2k), 0)$, and the y-axis at $E = (0, arsinh(2k))$.

[If necessary, we could compare the x-coordinates of A & D (and B & E), to confirm that $arsinh\left(k + \sqrt{k^2+1}\right) > arsinh(2k)$;

or equivalently that $k + \sqrt{k^2+1} > 2k \Leftrightarrow k^2 + 1 > k^2$

(as $k > 0$).]