STEP 2020, P3, Q12 - Solution (6 pages; 30/1/23)

(i) **Part 1**

X and Y are independent Geometric variables (so that eg

$P(X = x) = q^{x-1}p$)

and $P(S = s) = P(X + Y = s) = \sum_{k=1}^{s-1} P(X = k \;\&\; Y = s - k)$

$= \sum_{k=1}^{s-1} P(X = k).P(Y = s - k)$, as X & Y are independent

$= \sum_{k=1}^{s-1} q^{k-1}p.q^{s-k-1}p$

$= p^2q^{s-2}\sum_{k=1}^{s-1} 1$

$= (s-1)p^2q^{s-2}$ (for $s \geq 2$)

Part 2

$P(T \leq t) = 1 - P(T > t)$

$= 1 - P(at\ least\ one\ of\ X \;\&\; Y > t)$

$= 1 - [1 - P(neither\ X\ nor\ Y > t)]$

$= P(neither\ X\ nor\ Y > t)$

$= P(X \leq t\ and\ Y \leq t)$

$= P(X \leq t)P(Y \leq t)$, as as X & Y are independent

Now, $P(X \leq t) = \sum_{k=1}^{t} q^{k-1}p = \frac{p(q^t-1)}{1-q} = 1 - q^t$,

so that $P(T \leq t) = (1 - q^t)^2$

And $P(T = t) = P(T \leq t) - P(T \leq t - 1)$

$= (q^t - 1)^2 - (q^{t-1} - 1)^2$

$= [(q^t - 1) - (q^{t-1} - 1)][(q^t - 1) + (q^{t-1} - 1)]$

$= [q^t - q^{t-1}][q^t + q^{t-1} - 2]$

$= q^{t-1}(q - 1)(q^t + q^{t-1} - 2)$

$= q^{t-1}p(2 - q^t - q^{t-1})$, which is the same as the required expression

(ii) **Part 1**

Consider separately $U = 0$ and $U \geq 1$:

$P(U = 0) = \sum_{k=1}^{\infty} P(X = k \; and \; Y = k)$

$= \sum_{k=1}^{\infty} P(X = k)P(Y = k)$

$= \sum_{k=1}^{\infty} q^{k-1}p \,.\, q^{k-1}p$

$= \sum_{k=1}^{\infty} q^{2k-2}p^2$

$= \frac{p^2}{1-q^2} = \frac{p^2}{(1-q)(1+q)} = \frac{p}{1+q}$

For $U \geq 1$:

$P(U = u) = P(U \neq 0)$

$\times P(2nd \; person \; to \; obtain \; a \; Head \; obtains \; it \; u \; tosses$

$after \; the \; 1st \; person \; | U \neq 0)$

$= \left(1 - \frac{p}{1+q}\right) q^{u-1}p$

$= \frac{(1+q-p)q^{u-1}p}{1+q}$

$= \frac{2q.q^{u-1}p}{1+q}$

$= \frac{2q^u p}{1+q}$

Part 2

$P(W > w) = P(X > w \text{ } and \text{ } Y > w)$

$= P(X > w)P(Y > w)$

So $P(W \leq w) = 1 - (1 - P(X \leq w))(1 - P(Y \leq w))$

From (i) (Part 2),

$P(X \leq w)$ & $P(Y \leq w) = 1 - q^w$,

so that $P(W \leq w) = 1 - q^w . q^w = 1 - q^{2w}$

Then $P(W = w) = P(W \leq w) - P(W \leq w - 1)$

$= (1 - q^{2w}) - (1 - q^{2[w-1]})$

$= q^{2w-2} - q^{2w}$

[The Official sol'ns give $pq^{2w-2}(1 + q)$ as the answer,

and this can be rearranged as $(1 - q)q^{2w-2}(1 + q)$

$= q^{2w-2}(1 - q^2) = q^{2w-2} - q^{2w}$, as above.]

(iii) $P(S = 2 \text{ } and \text{ } T = 3) = P(X + Y = 2 \text{ } and \max(X, Y) = 3) = 0$

whilst $P(S = 2) \times P(T = 3)$

$= (2 - 1)p^2 q^{2-2} . q^{3-1} p(2 - q^3 + q^{3-1})$

$= p^2 q^2 p(2 - q^3 + q^2)$

and in general, $2 - q^3 + q^2 \neq 0$,

so $P(S = 2\ and\ T = 3) \neq P(S = 2) \times P(T = 3)$

(iv) **Part 1**

To show that U and W are independent, we need to establish that

$P(U = u\ and\ W = w) = P(U = u)P(W = w)$ (*)

When $U = 0$,

$P(U = u\ and\ W = w) = P(X = Y = w) = P(X = w)(Y = w)$

$= q^{w-1}p.q^{w-1}p = q^{2w-2}p^2$

and $P(U = u)P(W = w) = \frac{p}{1+q}\ (q^{2w-2} - q^{2w})$

$= \frac{pq^{2w-2}(1-q^2)}{1+q} = pq^{2w-2}(1 - q) = q^{2w-2}p^2$

So (*) holds when $U = 0$.

When $U \geq 1$,

$P(U = u\ and\ W = w) =$

$= P(X = w\ and\ Y = w + u\ OR\ Y = w\ and\ X = w + u)$

$= P(X = w\ and\ Y = w + u\ ) + P(Y = w\ and\ X = w + u)$

$= 2q^{w-1}p.q^{w+u-1}p$

$= 2q^{2w+u-2}p^2$

And $P(U = u)P(W = w) = \frac{2q^u p}{1+q}\ .(q^{2w-2} - q^{2w})$

$= \frac{2q^{2w+u-2}p(1-q^2)}{1+q}$

$= 2q^{2w+u-2}p(1 - q)$

$= 2q^{2w+u-2}p^2$

So (*) holds when $U \geq 1$ as well.

Hence U and W are independent.

Part 2

From (iii), we have established that S and T are not independent.

We need to show that the following pairs of variables are also not independent:

(a) S and U (b) S and W (c) T and U (d) T and W

(a) S and U

$U = 0$:

$P(S = s\ and\ U = 0) = P\left(X = \frac{s}{2}\ \&\ Y = \frac{s}{2}\right) = P(X = \frac{s}{2})P(Y = \frac{s}{2})$,

which is zero if s is odd;

whereas $P(S = s)P(U = 0)$ is non-zero for odd s

So S and U are not independent. [No need to investigate $U \geq 1$.]

(b) S and W

$P(W = 1)$ will vary with knowledge of the value of S: If $S = 2$, for example, $P(W = 1) = 1$, as both X and Y must be 1, and when $S \geq 4$, $P(W = 1) \neq 1$ (as $X = Y = 2$ is possible, for example, when $S = 4$).

And so S and W are not independent.

(c) T and U

[T is Max(X,Y); U is $|X - Y|$]

The knowledge that U is large increases the probability that T is large, as it rules out situations where X & Y are both small.

So T and U are not independent.

(d) T and W

[T is Max(X,Y); W is Min(X,Y)]

If eg it is known that $T \leq 10$ (so that both X & Y ≤ 10) then

$W > 10$ is not possible.

So T and W are not independent.