STEP 2020, P3, Q1 - Solution (2 pages; 5/7/21)

(i) By Parts,

$I(a,b) = \left[\frac{1}{b} sinbxcos^a x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{b} sinbx.acos^{a-1}x(-sinx)dx$

Now, $\cos(b-1)\,x = cosbxcosx + sinbxsinx$ (valid when b is an integer ≥ 1), so that

$I(a,b) = 0 + \frac{a}{b}\{I(a-1,b-1) - \int_0^{\frac{\pi}{2}} cos^{a-1}xcosbxcosxdx\}$

$= \frac{a}{b}\{I(a-1,b-1) - I(a,b)\}$

$\Rightarrow bI(a,b) = aI(a-1,b-1) - aI(a,b)$

$\Rightarrow (a+b)I(a,b) = aI(a-1,b-1)$

$\Rightarrow I(a,b) = \frac{a}{a+b} I(a-1,b-1)$, as required.

(ii) Let $P(n)$ be the proposition in the question.

To prove that $P(0)$ is true:

LHS $= \int_0^{\frac{\pi}{2}} cos(2m+1)x\ dx = \left[\frac{1}{2m+1} \sin\,(2m+1)x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2m+1}$ if m is even, and $-\frac{1}{2m+1}$ if m is odd

And RHS $= (-1)^m \frac{(2m)!m!}{m!(2m+1)!}$

$= \frac{1}{2m+1}$ if m is even, and $-\frac{1}{2m+1}$ if m is odd

Thus $P(0)$ is true.

Assume that $P(k)$ is true, so that

$\int_0^{\frac{\pi}{2}} cos^k x cos(k+2m+1)x\,dx = (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!}$ (A)

We wish to prove that $P(k+1)$ is then true; ie that

$\int_0^{\frac{\pi}{2}} cos^{k+1} x cos(k+1+2m+1)x\,dx$

$= (-1)^m \frac{2^{k+1}(k+1)!(2m)!(k+1+m)!}{m!(2k+2+2m+1)!}$ (B)

Writing $a = k+1$ & $b = k+2m+2$,

LHS of (B) $= \frac{k+1}{2k+2m+3} \int_0^{\frac{\pi}{2}} cos^k x cos(k+2m+1)x\,dx$

$= \frac{k+1}{2k+2m+3} (-1)^m \frac{2^k k!(2m)!(k+m)!}{m!(2k+2m+1)!}$, from (A)

$= (-1)^m \frac{2^k (k+1)!(2m)!(k+m)!(2k+2m+2)}{m!(2k+2m+3)!}$

$= (-1)^m \frac{2^{k+1}(k+1)!(2m)!(k+m+1)!}{m!(2k+2m+3)!}$; ie (B)

So if $P(k)$ is true, then $P(k+1)$ is true.

As $P(0)$ is true, it follows that $P(1), P(2)$, ... are true, and hence $P(n)$ is true for all non-negative n, by induction.