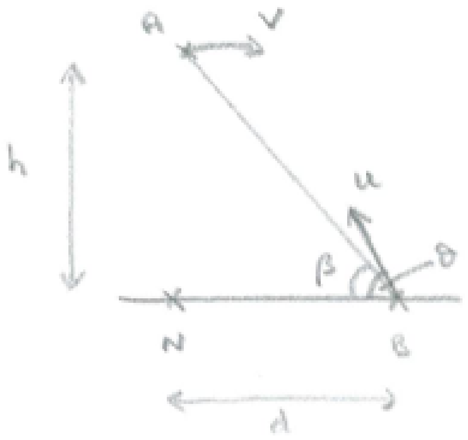
STEP 2020, P2, Q9 - Solution (4 pages; 6/7/21)

1st part



Let (x, y) be the coordinates of the point of collision, where N is the Origin.

[The following approach doesn't lead anywhere:

Then, from the Cartesian form of the eq'n of a projectile,

for A: $y = h - \frac{gx^2}{2V^2}$, and for B: $y = (d - x)tan\theta - \frac{g(d-x)^2}{2U^2cos^2\theta}$

The resulting quadratic in x is to have one sol'n.

So $h - \frac{gx^2}{2V^2} = (d - x)tan\theta - \frac{g(d-x)^2}{2U^2cos^2\theta}$,

but the expression for the discriminant is too complicated to be worth pursuing.]

If T is the time at collision, then equating the x & y coordinates for A & B:

$x = VT = d - Ucos\theta.T$ (1),

and $y = h - \frac{g}{2}T^2 = Usin\theta.T - \frac{g}{2}T^2$ (2)

Then (1) $\Rightarrow T(V + Ucos\theta) = d$ and (2) $\Rightarrow h = Usin\theta.T$, and eliminating T gives $\frac{d}{V+Ucos\theta} = \frac{h}{Usin\theta}$,

so that $dUsin\theta = hV + hUcos\theta$

$\Rightarrow dsin\theta - hcos\theta = \frac{Vh}{U}$, as required.

[This question is a good example of where writing something down (eq'n (2)), perhaps without knowing where it will lead, leads to spotting a useful device – ie that the $\frac{g}{2}T^2$ will cancel.]

(i) $tan\beta = \frac{h}{d}$

From the 1st part, $dtan\theta - h = \frac{Vh}{U}sec\theta$ (3)

Result to prove: $\theta > \beta \Leftrightarrow tan\theta > tan\beta$ [as both angles are acute]

$\Leftrightarrow tan\theta - \frac{h}{d} > 0$

$\Leftrightarrow \frac{dtan\theta - h}{d} > 0$ [as $d > 0$]

$\Leftrightarrow dtan\theta - h > 0$ [as $d > 0$],

which follows from (3), as each of V, U, h & $sec\theta$ is positive.

So we have established that $\theta > \beta$.

(ii) As the y coordinate will be positive when the collision takes place, $h - \frac{g}{2}T^2 \geq 0$, from (2).

Then, as $h = Usin\theta.T$ (established from (2)),

$h - \frac{g}{2}(\frac{h}{Usin\theta})^2 \geq 0$

$\Rightarrow 1 \geq \frac{gh}{2U^2sin^2\theta}$

$\Rightarrow U^2 sin^2\theta \geq \frac{gh}{2}$

$\Rightarrow Usin\theta \geq \sqrt{\frac{gh}{2}}$ [as $Usin\theta > 0$], as required.

(iii) From the 1st part, $dsin\theta - hcos\theta = \frac{Vh}{U}$

$\Leftrightarrow \frac{U}{V} = \frac{h}{dsin\theta - hcos\theta}$

Also, $tan\beta = \frac{h}{d}$, and so $\frac{U}{V} = \frac{tan\beta}{sin\theta - tan\beta cos\theta} = \frac{sin\beta}{sin\theta cos\beta - sin\beta cos\theta}$

$= \frac{sin\beta}{\sin(\theta - \beta)} > sin\beta$, as required

- as $\sin(\theta - \beta) \leq 1$ and $\sin(\theta - \beta) \neq 1$ (otherwise $\theta - \beta = \frac{\pi}{2}$, which isn't possible, as both θ & β are acute).

Final part

The vertical speed of B at the point of collision is $Usin\theta - gT$

And the y-coordinate of the point of collision is $h - \frac{g}{2}T^2$, from (2).

So the result to prove is:

$h - \frac{g}{2}T^2 > \frac{h}{2} \Leftrightarrow Usin\theta - gT > 0$

ie $\frac{h}{2} > \frac{g}{2}T^2$ or $T < \sqrt{\frac{h}{g}} \Leftrightarrow T < \frac{Usin\theta}{g}$ (4)

Now $h = Usin\theta.T$, from (2),

and so (4) can be rewritten as

$\frac{h}{Usin\theta} < \sqrt{\frac{h}{g}} \Leftrightarrow \frac{h}{Usin\theta} < \frac{Usin\theta}{g}$

or $\frac{h^2}{U^2sin^2\theta} < \frac{h}{g} \Leftrightarrow gh < U^2sin^2\theta$ (as all the quantities are positive)

and the left-hand inequality can be seen to be equivalent to the right-hand one.