STEP 2020, P2, Q6 - Solution (5 pages; 1/7/21)

(i) [The columns of a matrix usually have more significance than the rows, and so $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ would generally be preferred.]

$M^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

and so $tr(M^2) = a^2 + 2bc + d^2$

And $[tr(M)]^2 - 2\det(M) = (a+d)^2 - 2(ad-bc)$

$= a^2 + d^2 + 2bc$

Thus, $tr(M^2) = [tr(M)]^2 - 2\det(M)$, as required.

(ii) **1st part**

Suppose that $M^2 = I$, but $M \neq \pm I$

Then $M^{-1} = M$, so that $\frac{1}{\det(M)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (1),

so that $\frac{-c}{\det(M)} = c$, and hence either $\det(M) = -1$ or $c = 0$. (2)

If $c = 0$, then $\det(M) = ad$,

and so, from (1), $\frac{d}{ad} = a \Rightarrow a = \pm 1$

Also $\frac{a}{ad} = d \Rightarrow d = \pm 1$,

and $\frac{-b}{ad} = b \Rightarrow$ either $ad = 1$ or $b = 0$

If $b = 0$ (and $c = 0$), then (as $M \neq \pm I$),

either $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

In both cases, $tr(M) = 0$, and $\det(M) = -1$.

If $c \neq 0$, so that $\det(M) = -1$ (from (2)):

$tr(I) = [tr(M)]^2 - 2\det(M)$, from (i),

so that $[tr(M)]^2 = 2 + 2(-1) = 0$, and hence $tr(M) = 0$

Thus, if $\boldsymbol{M^2 = I}$, but $\boldsymbol{M \neq \pm I}$, then $\boldsymbol{tr(M) = 0}$ and $\boldsymbol{\det(M) = -1}$.

Suppose that $tr(M) = 0$ and $\det(M) = -1$,

so that we can write $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and $-a^2 - bc = -1$,

or $a^2 + bc = 1$

Then $M^2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

And $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \neq \pm I$, as the elements on the leading diagonal cannot be equal unless $a = 0$, in which case $M \neq \pm I$.

Thus, if $\boldsymbol{tr(M) = 0}$ and $\boldsymbol{\det(M) = -1}$, then $\boldsymbol{M^2 = I}$, but $\boldsymbol{M \neq \pm I}$.

[It doesn't matter whether we say "but" or "and".]

And so, $M^2 = I$, but $M \neq \pm I \Leftrightarrow tr(M) = 0$ and $\det(M) = -1$, as required.

2nd part

Suppose that $M^2 = -I$.

Then $M^{-1} = -M$, so that $\frac{1}{\det(M)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (3),

so that $\frac{-c}{\det(M)} = -c$, and hence either $\det(M) = 1$ or $c = 0$. (4)

If $c = 0$, then $\det(M) = ad$,

and so, from (3), $\frac{d}{ad} = -a \Rightarrow a^2 = -1$, which isn't possible, as a is real.

So $c \neq 0$, and hence $\det(M) = 1$.

From (i), $tr(-I) = [tr(M)]^2 - 2\det(M)$,

so that $[tr(M)]^2 = -2 + 2(1) = 0$, and hence $tr(M) = 0$

Thus, if $\boldsymbol{M^2 = -I}$, then $\boldsymbol{tr(M) = 0}$ and $\boldsymbol{\det(M) = 1}$.

Suppose that $tr(M) = 0$ and $\det(M) = 1$,

so that we can write $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and $-a^2 - bc = 1$,

or $a^2 + bc = -1$

Then $M^2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$

Thus, if $\boldsymbol{tr(M) = 0}$ and $\boldsymbol{\det(M) = 1}$, then $\boldsymbol{M^2 = -I}$.

And so, $M^2 = -I \Leftrightarrow tr(M) = 0$ and $\det(M) = 1$, as required.

(iii) **1st part**

First of all, $M^2 = \pm I \Rightarrow M^4 = I^2 = I$ or $M^4 = (-I)^2 = I$

Let $N = M^2$. Result to prove: $N^2 = I \Rightarrow N = \pm I$

Suppose that $N^2 = I$ but $N \neq \pm I$ (*)

Then, from (ii), as the elements of $N = M^2$ will be real,

$\det(N) = -1$

And $N = M^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

So $a^2 + 2bc + d^2 = 0$

and $(a^2+bc)(bc+d^2) - (ac+cd)(ab+bd) = -1$

$\Rightarrow a^2bc + a^2d^2 + b^2c^2 + bcd^2 - a^2bc - 2abcd - bcd^2 = -1$

$\Rightarrow a^2d^2 + b^2c^2 - 2abcd = -1$

$\Rightarrow (ad-bc)^2 = -1$, which isn't possible, contradicting (*).

Hence $N^2 = I \Rightarrow N = \pm I$,

and so $M^4 = I \Leftrightarrow M^2 = \pm I$, as required.

2nd part

Let $N = M^2$ again. Then, from (ii),

$M^4 = -I \; or \; N^2 = -I \Leftrightarrow tr(N) = 0 \; and \; \det(N) = 1$,

ie $tr(M^2) = 0 \; and \; \det(M^2) = 1$

Then, $\det(M^2) = 1 \Leftrightarrow [det(M)]^2 = 1 \Leftrightarrow det(M) = \pm 1$,

and, from (i), $[tr(M)]^2 - 2\det(M) = 0$,

$\Leftrightarrow [tr(M)]^2 = 2$, and $det(M) = +1$ only.

Thus, the required necessary and sufficient conditions are

that $det(M) = 1$ and $tr(M) = \pm\sqrt{2}$

(iv) Let $M = \begin{pmatrix} \sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$

Then $det(M) = 1$ and $tr(M) = \sqrt{2}$, so that, from the 2nd part of (iii), $M^4 = -I$, and hence $M^8 = I$.

M is not of the form $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$, and so is not a rotation;

and $\begin{pmatrix}0\\1\end{pmatrix}$ maps to $\begin{pmatrix}1\\0\end{pmatrix}$, implying a reflection in $y = x$ (were M to represent a reflection), which is contradicted by the image of $\begin{pmatrix}1\\0\end{pmatrix}$ being $\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$.

Thus $M = \begin{pmatrix}\sqrt{2} & 1\\-1 & 0\end{pmatrix}$ is a suitable example.