STEP 2020, P2, Q1 - Solution (2 pages; 4/6/21)

(i) Writing $x = \frac{1}{1-u}$, so that $x - 1 = \frac{1-(1-u)}{1-u} = \frac{u}{1-u}$,

and $dx = (1-u)^{-2}du$,

$I = \int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}} dx = \int \frac{(1-u)^{-2}}{(1-u)^{-\frac{3}{2}}(\frac{u}{1-u})^{\frac{1}{2}}}\ du = \int u^{-\frac{1}{2}}\, du = \frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$

Then $x = \frac{1}{1-u} \Rightarrow 1 - u = \frac{1}{x} \Rightarrow u = 1 - \frac{1}{x} = \frac{x-1}{x}$,

and so $I = 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + C$

(ii) Let $v = x - 2$,

so that $J = \int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\ dx = \int \frac{1}{v^{\frac{3}{2}}(v+3)^{\frac{1}{2}}} dv$

Then consider the substitution $v = \frac{1}{1-u}$ again,

so that $v + 3 = \frac{1+3(1-u)}{1-u} = \frac{4-3u}{1-u}$ and $dv = (1-u)^{-2}du$

Then $J = \int \frac{(1-u)^{-2}}{(1-u)^{-\frac{3}{2}}\left(\frac{4-3u}{1-u}\right)^{\frac{1}{2}}} du = \int (4-3u)^{-\frac{1}{2}}\, du$

$= \frac{(4-3u)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(-3)} + D$

As $x = v + 2 = \frac{1}{1-u} + 2$,

$1 - u = \frac{1}{x-2}$, and $u = 1 - \frac{1}{x-2} = \frac{x-3}{x-2}$,

so that $J = -\frac{2}{3}(4-3u)^{\frac{1}{2}} + D$

$= -\frac{2}{3}\left(4 - 3(\frac{x-3}{x-2})\right)^{\frac{1}{2}} + D$

$= -\frac{2}{3}\left(\frac{4x-8-3x+9}{x-2}\right)^{\frac{1}{2}} + D$

$= -\frac{2}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + D$

(iii) [$\frac{\pi}{3}$ suggests an integral of the form $\int \frac{1}{\sqrt{a^2-x^2}}dx$, and we see that we obtained the integrand $(4-3u)^{-\frac{1}{2}}$ in (ii), suggesting that the same approach in (iii) might produce an integrand of the form$(c-(u-d)^2)^{-\frac{1}{2}}$, if we're lucky.]

Let $v = x - 1$, so that $3x - 2 = 3(v+1) - 2 = 3v + 1$,

and $K = \int_2^\infty \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}dx = \int_1^\infty \frac{1}{v(v-1)^{\frac{1}{2}}(3v+1)^{\frac{1}{2}}}\ dv$

Then let $v = \frac{1}{1-u}$ once again,

But now $v - 1 = \frac{1-(1-u)}{1-u} = \frac{u}{1-u}$, which won't lead to the required form (a 3 term quadratic in u, arising from $v - 1$ & $3v + 1$).

However, $v = \frac{2}{1-u}$ gives $v - 1 = \frac{2-(1-u)}{1-u} = \frac{1+u}{1-u}$,

and $3v + 1 = \frac{6+(1-u)}{1-u} = \frac{7-u}{1-u}$; and $dv = 2(1-u)^{-2}du$

Also, $1 - u = \frac{2}{v}$, so that $u = 1 - \frac{2}{v}$

Then $K = \int_{-1}^1 \frac{2(1-u)^{-2}}{(\frac{2}{1-u})\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}\left(\frac{7-u}{1-u}\right)^{\frac{1}{2}}}\ du = \int_{-1}^1 \frac{1}{\sqrt{(1+u)(7-u)}}du$

And $(1+u)(7-u) = 7 + 6u - u^2 = 16 - (u-3)^2$,

so that $K = \left[\arcsin\left(\frac{u-3}{4}\right)\right]_{-1}^{1} = \arcsin\left(-\frac{1}{2}\right) - \arcsin(-1)$

$= -\frac{\pi}{6} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{3}$, as required.