STEP 2020, P2, Q11 - Solution (4 pages; 12/5/21)

[Probably too long to be attempted fully in the STEP exam.]

(i) **1st part**

P(game never ends) = P(HTHTHT …) + P(THTHTH …)

$= pqpq \dots + qpqp \dots . = 2\lim_{n\to\infty} p^n q^n = 0$

2nd part

P(A wins |1st toss is H)

= P(2nd toss is H)

$+ \sum_{r=1}^{\infty}$ P(2nd toss is T *and* A wins on (2r + 2)nd toss

|1st toss is H)

$= p + \sum_{r=1}^{\infty} q(pq)^{r-1}p^2$

$= p + qp^2 . \frac{1}{1-pq}$

$= \frac{p(1-pq)+qp^2}{1-pq}$

$= \frac{p}{1-pq}$, as required.

3rd part

By symmetry, P(B wins |1st toss is T) $= \frac{q}{1-qp}$

And P(A wins |1st toss is T) = 1 − P(B wins |1st toss is T)

$= 1 - \frac{q}{1-qp} = \frac{1-qp-q}{1-qp} = \frac{p-qp}{1-qp} = \frac{p(1-q)}{1-qp} = \frac{p^2}{1-qp}$

[Alternatively, P(A wins |1st toss is T)

= P(2nd toss is H). P(A wins |1st toss is H)

$= p . \frac{p}{1-pq} = \frac{p^2}{1-qp}$]

Then $P(A\ wins) = p.$P(A wins |1st toss is H)

$+q.$P(A wins |1st toss is T)

$= p.\frac{p}{1-pq} + q.\frac{p^2}{1-qp}$

$= \frac{p^2(1+q)}{1-pq}$

[Check: By symmetry, $P(B\ wins) = \frac{q^2(1+p)}{1-qp}$

and $\frac{p^2(1+q)}{1-pq} + \frac{q^2(1+p)}{1-qp} = \frac{p^2+p^2q+q^2+q^2p}{1-pq} = \frac{p^2+q^2+pq(p+q)}{1-pq}$

$= \frac{p^2+q^2+pq}{1-pq} = \frac{p(p+q)+q^2}{1-pq} = \frac{p+q^2}{1-pq} = \frac{1-q+q^2}{1-pq} = \frac{1-q(1-q)}{1-pq} = \frac{1-qp}{1-pq} = 1$]

(ii) **1st part**

P(A wins |1st toss is H) $= P(2nd\ \&\ 3rd\ tosses\ are\ H)$

+P(2nd toss is T). P(A wins |1st toss is T)

+P(2nd toss is H). P(3rd toss is T). P(A wins |1st toss is T)

$= p^2 + (q + pq)$P(A wins |1st toss is T) , as required.

2nd part

P(A wins |1st toss is T)

$= P(2nd\ toss\ is\ T).$ P(A wins |1st two tosses are T)

$+P(2nd\ toss\ is\ H).P(A\ wins|$1st toss is H)

$= q.P(3rd\ toss\ is\ H).P($P(A wins |1st toss is H)

$+p.$P(A wins |1st toss is H)

$= (qp + p)$P(A wins |1st toss is H)

or $(p + pq)$P(A wins |1st toss is H)

3rd part

$P(A\ wins) = p.\mathrm{P(A\ wins\ |1st\ toss\ is\ H)}$

$+q.\mathrm{P(A\ wins\ |1st\ toss\ is\ T)}$

$= ph + qt$, say (1)

And from the 1st and 2nd parts,

$h = p^2 + (q + pq)t$ and $t = (p + pq)h$,

so that $h = p^2 + (q + pq)(p + pq)h$,

and hence $h\{1 - q(1 + p)p(1 + q)\} = p^2$

Then (1) becomes

$ph + q(p + pq)h = \frac{p^2\{p+q(p+pq)\}}{1-q(1+p)p(1+q)}$

Now, $q(1 + p)p(1 + q) = (1 - p)(1 + p)(1 - q)(1 + q)$

$= (1 - p^2)(1 - q^2)$

and so we just need to show that

$p + q(p + pq) = 1 - q^3$

Using the result that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$,

$1 - q^3 = (1 - q)(1 + q + q^2) = p(1 + q + q^2)$

$= p + q(p + pq)$, as required.

(iii) **1st part**

$\mathrm{P(A\ wins\ |1st\ toss\ is\ H)} = P(next\ (a - 1)\ tosses\ are\ H)$

$+ \sum_{r=0}^{a-2}\{P(next\ r\ tosses\ are\ H\ and\ then\ there\ is\ a\ T).$

$\mathrm{P(A\ wins\ |1st\ toss\ is\ T)}\}$

which can be written as:

$h = p^{a-1} + \sum_{r=0}^{a-2} p^r qt = p^{a-1} + qt\,.\frac{1-p^{a-1}}{1-p}$

$= p^{a-1} + t(1 - p^{a-1})$ (2)

And P(A wins |1st toss is T)

$= \sum_{r=0}^{b-2}\{P(next\ r\ tosses\ are\ T\ and\ then\ there\ is\ an\ H).$

P(A wins |1st toss is H)}

so that $t = \sum_{r=0}^{b-2} q^r ph = ph\,.\frac{1-q^{b-1}}{1-q} = h(1 - q^{b-1})$

Then, substituting into (2),

$h = p^{a-1} + h(1 - q^{b-1})(1 - p^{a-1})$

so that $h\{1 - (1 - q^{b-1})(1 - p^{a-1})\} = p^{a-1}$

and $h = \frac{p^{a-1}}{1-(1-q^{b-1})(1-p^{a-1})}$

Then $P(A\ wins) = ph + qt$

$= \frac{p^a}{1-(1-q^{b-1})(1-p^{a-1})} + \frac{qp^{a-1}(1-q^{b-1})}{1-(1-q^{b-1})(1-p^{a-1})}$

$= \frac{p^{a-1}\{p+q(1-q^{b-1})\}}{1-(1-q^{b-1})(1-p^{a-1})}$

$= \frac{p^{a-1}\{1-q^b\}}{1-(1-q^{b-1})(1-p^{a-1})}$

2nd part

When $a = b = 2$, $P(A\ wins) = \frac{p\{1-q^2\}}{1-(1-q)(1-p)} = \frac{p\{1-q^2\}}{p+q-pq}$

$= \frac{p(1-q)(1+q)}{1-pq} = \frac{p^2(1+q)}{1-pq}$, which is the result from part (i).