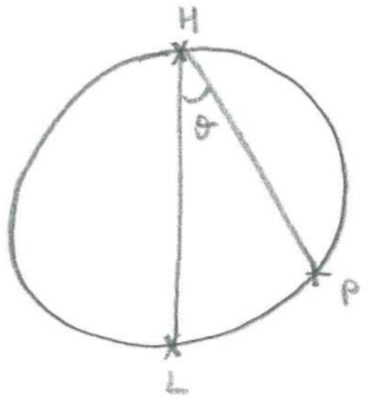
STEP 2020, P2, Q10 - Solution (4 pages; 28/5/21)

(i) **1st part**



[See the Official Sol'ns for a quicker approach, where the tangential force is set equal to zero. Note that the normal reaction does no work (being perpendicular to the direction of motion), and so does not feature in the following energy method.]

The potential energy, P of the particle when it is at rest in the position shown in the diagram is:

gravitational PE + elastic PE

$= mg(2a - HPcos\theta) + \frac{1}{2}\left(\frac{\lambda}{l}\right)(HP - l)^2$,

where the gravitational PE is measured relative to the level of L.

As the triangle PHL is right-angled (as HL is a diameter),

$HP = HLcos\theta = 2acos\theta$

and so $P = 2amg(1 - cos^2\theta) + \frac{1}{2}\left(\frac{\lambda}{l}\right)(2acos\theta - l)^2$

We wish to show that, when P has a minimum, $cos\theta = \frac{\lambda l}{2(a\lambda - mgl)}$

[If the particle is stationary and the potential energy is at a minimum, then any increase in the potential energy would mean a reduction in kinetic energy, which isn't possible, and so the particle must be in (stable) equilibrium.]

$\frac{dP}{d\theta} = 2amg(-2cos\theta)(-sin\theta) + \frac{\lambda}{2l} \cdot 2(2acos\theta - l)(-2asin\theta)$

$= 2asin\theta\{2mgcos\theta - \frac{\lambda}{l}(2acos\theta - l)\}$

so that $\frac{dP}{d\theta} = 0 \Rightarrow sin\theta = 0$ or

$2mgcos\theta - \frac{\lambda}{l}(2acos\theta - l) = 0$

$\Rightarrow cos\theta\left(2mg - \frac{2\lambda a}{l}\right) = -\lambda$

$\Rightarrow cos\theta = \frac{\lambda l}{2(\lambda a - mgl)}$

Consider the sign of $\frac{d^2P}{d\theta^2}$:

$\frac{dP}{d\theta} = 2asin\theta\{2mgcos\theta - \frac{\lambda}{l}(2acos\theta - l)\}$

$= 2amgsin(2\theta) - \frac{2a^2\lambda}{l}\sin(2\theta) + 2a\lambda sin\theta$

so that $\frac{d^2P}{d\theta^2} = 4amgcos(2\theta) - \frac{4a^2\lambda}{l}\cos(2\theta) + 2a\lambda cos\theta$

$= 4a(mg - \frac{a\lambda}{l})(2cos^2\theta - 1) + 2a\lambda cos\theta$

When $cos\theta = \frac{\lambda l}{2(\lambda a - mgl)}$,

$\frac{d^2P}{d\theta^2} = \frac{4a}{l}\left(-\frac{\lambda l}{2cos\theta}\right)(2cos^2\theta - 1) + 2a\lambda cos\theta$

$= \frac{2a\lambda}{cos\theta}(1 - 2cos^2\theta + cos^2\theta)$

$= \frac{2a\lambda sin^2\theta}{cos\theta}$

Noting from the diagram that $0 \leq \theta \leq \frac{\pi}{2}$, but assuming that $\alpha \neq \frac{\pi}{2}$ [though there doesn't seem to be anything preventing the equilibrium position being at H (though it wouldn't be stable)],

and given that $\alpha > 0$, it follows that $\frac{d^2P}{d\theta^2} > 0$ when $\theta = \alpha$,

and so a minimum occurs at $\theta = \alpha$, as required.

2nd part

As $cos\alpha < 1$, it follows that $\frac{\lambda l}{2(\lambda a - mgl)} < 1$,

so that $\lambda l < 2(\lambda a - mgl)$ $(cos\alpha > 0 \Rightarrow \lambda a - mgl > 0)$

$\Rightarrow \lambda(l - 2a) < -2mgl$

$\Rightarrow \lambda > \frac{-2mgl}{l-2a}$ (as $l < 2a$)

ie $\lambda > \frac{2mgl}{2a-l}$, as required.

(ii) **1st part**

By conservation of energy,

$\frac{1}{2}mu^2 + \frac{1}{2}\left(\frac{\lambda}{l}\right)(2a - l)^2$

$= 2amg(1 - cos^2\beta) + \frac{1}{2}\left(\frac{\lambda}{l}\right)(2acos\beta - l)^2$

(from the 1st part of (i))

$\Rightarrow \frac{1}{2}mu^2 + \frac{2a^2\lambda}{l} + \frac{\lambda l}{2} - 2a\lambda$

$= 2amg - 2amgcos^2\beta + \frac{2a^2\lambda}{l}cos^2\beta + \frac{\lambda l}{2} - 2a\lambda cos\beta$

$\Rightarrow \frac{1}{2}mu^2 + \frac{2a^2\lambda}{l} - 2a\lambda$

$= 2amg + cos^2\beta(\frac{2a^2\lambda}{l} - 2amg) - 2a\lambda cos\beta$ (1)

Now, $cos\alpha = \frac{\lambda l}{2(\lambda a - mgl)} \Rightarrow \frac{2a^2\lambda}{l} - 2amg = \frac{2a}{l}(a\lambda - mgl)$

$= \frac{\lambda a}{cos\alpha}$

Then (1) becomes

$\frac{1}{2}mu^2 + \frac{\lambda a}{cos\alpha} - 2a\lambda = cos^2\beta . \frac{\lambda a}{cos\alpha} - 2a\lambda cos\beta$

$\Rightarrow \frac{mu^2 cos\alpha}{2a\lambda} + (1 - cos^2\beta) - 2cos\alpha(1 - cos\beta) = 0$

$\Rightarrow \frac{mu^2 cos\alpha}{2a\lambda} + (1 - cos^2\beta) - 2cos\alpha(1 - cos\beta) = 0$

$\Rightarrow (1 - cos\alpha)^2 + \frac{mu^2 \boldsymbol{cos\alpha}}{2a\lambda}$

$= (1 - cos\alpha)^2 + 2cos\alpha(1 - cos\beta) - (1 - cos^2\beta)$

$= 1 - 2cos\alpha + cos^2\alpha + 2cos\alpha - 2cos\alpha cos\beta - 1 + cos^2\beta$

$= cos^2\alpha - 2cos\alpha cos\beta + cos^2\beta$

$= (cos\alpha - cos\beta)^2$, as required.

2nd part

Given that $(1 - cos\alpha)^2 + \frac{mu^2 cos\alpha}{2a\lambda} = (cos\alpha - cos\beta)^2$,

$\frac{mu^2 cos\alpha}{2a\lambda} = (cos\alpha - cos\beta)^2 - (1 - cos\alpha)^2$

and as $cos\alpha > 0$ & $cos\beta > 0$,

$\frac{mu^2 cos\alpha}{2a\lambda} < cos^2\alpha - (1 - cos\alpha)^2 = 2cos\alpha - 1$,

so that $u^2 < \frac{2a\lambda}{m}(2 - sec\alpha)$, as required.