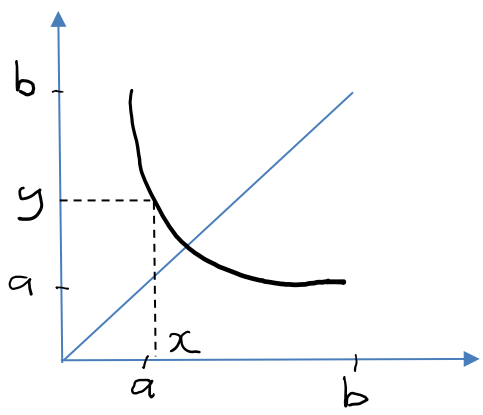
STEP 2020, P3, Q11 - Solution (4 pages; 4/6/23)

(i) 1st Part

Noting that the 3rd condition means that f has reflective symmetry in $y = x$, an example of f is shown below:



Then, for $y \in [a, b]$, as f is decreasing,

$$P(Y \leq y) = P(X \geq x) = \frac{b-x}{b-a} = \frac{b-f^{-1}(y)}{b-a}$$

$= \frac{b-f(y)}{b-a}$, as required.

2nd Part

$f_Y(y) = F'_Y(y)$, where $F_Y(y) = P(Y \leq y)$ [the cumulative distribution function of Y]

So $f_Y(y) = \frac{d}{dy}\left[\frac{b-f(y)}{b-a}\right] = -\frac{1}{b-a}f'(y)$ for $y \in [a, b]$ (and zero elsewhere).

3rd Part

$E(Y^2) = \int_a^b y^2 f_Y(y)\, dy$

$= -\frac{1}{b-a}\int_a^b y^2 f'(y)\, dy$

$= -\frac{1}{b-a}[y^2 f(y)]_a^b + \frac{1}{b-a}\int_a^b 2yf(y)\, dy$ (by Parts)

$= -\frac{1}{b-a}(b^2 a - a^2 b) + \int_a^b \frac{2xf(x)}{b-a}\, dx$

$= -ab + \int_a^b \frac{2xf(x)}{b-a}\, dx$, as required.

(ii) When $X = a, Z = \left(\frac{1}{c} - \frac{1}{a}\right)^{-1} = b$, and when $X = b, Z = a$.

Also, Z decreases as X increases.

And, if $z = f(x) = \left(\frac{1}{c} - \frac{1}{x}\right)^{-1}$, then as $x = \left(\frac{1}{c} - \frac{1}{z}\right)^{-1}$,

$f^{-1}(z) = f^{-1}\big(f(x)\big) = x = f(z)$, so that $f(x) = f^{-1}(x)$

Thus, f satisfies the conditions in (i), and so the results established for Y will apply to Z.

So $E(Z) = \int_a^b z f_Z(z)\, dz$

$= -\frac{1}{b-a}\int_a^b z f'(z)\, dz$,

where $f'(z) = \frac{d}{dz}\left(\frac{1}{c} - \frac{1}{z}\right)^{-1} = -\left(\frac{1}{c} - \frac{1}{z}\right)^{-2}(z^{-2})$

$= -\left(\frac{z}{c} - 1\right)^{-2} = -c^2(z-c)^{-2}$

and hence $E(Z) = \frac{c^2}{b-a}\int_a^b z(z-c)^{-2}\,dz$

$= \frac{c^2}{b-a}\int_a^b \frac{z-c}{(z-c)^2} + \frac{c}{(z-c)^2}\,dz$

$= \frac{c^2}{b-a}[\ln|z-c| - \frac{c}{z-c}]_a^b$

$= \frac{c^2}{b-a}(\ln\left|\frac{b-c}{a-c}\right| - \frac{c}{b-c} + \frac{c}{a-c})$

Now $\frac{1}{c} = \frac{1}{a} + \frac{1}{b} \Rightarrow \frac{1}{c} - \frac{1}{a} = \frac{1}{b}$, so that $\frac{a-c}{ac} = \frac{1}{b}$

and hence $\frac{c}{a-c} = \frac{b}{a}$, and by symmetry $\frac{c}{b-c} = \frac{a}{b}$ (**)

So $E(Z) = \frac{c^2}{b-a}(\ln\left|\frac{b-c}{a-c}\right| - \frac{a}{b} + \frac{b}{a})$

[In fact, $\frac{c^2}{b-a}\left(-\frac{a}{b} + \frac{b}{a}\right) = \frac{c^2}{b-a}\cdot\frac{b^2-a^2}{ab} = \frac{c^2(b+a)}{ab}$,

and as $c = \frac{1}{(\frac{1}{a}+\frac{1}{b})} = \frac{ab}{a+b}$, it simplifies further to c]

Also, from the 3rd Part of (i), $E(Z^2) = -ab + \int_a^b \frac{2zf(z)}{b-a}\,dz$

$= -ab + \frac{2}{b-a}\int_a^b z\left(\frac{1}{c} - \frac{1}{z}\right)^{-1} dz$

$= -ab + \frac{2c}{b-a}\int_a^b \frac{z^2}{z-c}\,dz$

$= -ab + \frac{2c}{b-a}\int_a^b \frac{z^2-c^2}{z-c} + \frac{c^2}{z-c}\,dz$

$= -ab + \frac{2c}{b-a}[\frac{1}{2}z^2 + cz + c^2\ln|z-c|]_a^b$

$= -ab + \frac{2c}{b-a}(\frac{1}{2}(b^2-a^2) + c(b-a) + c^2\ln\left|\frac{b-c}{a-c}\right|)$

$= -ab + c(b+a) + 2c^2 + \frac{2c^3}{b-a}\ln\left|\frac{b-c}{a-c}\right|$

As $\frac{c}{b-c} = \frac{a}{b}$, $bc = ab - ac$, so that $-ab + c(b+a) = 0$ (*)

and so, writing $A = \ln\left|\frac{b-c}{a-c}\right|$, $E(Z^2) = 2c^2 + \frac{2c^3 A}{b-a}$

Then $VarZ = E(Z^2) - [E(Z)]^2$

$= 2c^2 + \frac{2c^3 A}{b-a} - (\frac{c^2}{b-a}(A - \frac{a}{b} + \frac{b}{a}))^2$

Now, $-\frac{a}{b} + \frac{b}{a} = \frac{b^2-a^2}{ab} = \frac{(b-a)(b+a)}{c(b+a)}$ (from (*))

$= \frac{b-a}{c}$

so that $VarZ = 2c^2 + \frac{2c^3 A}{b-a} - \frac{c^4}{(b-a)^2}(A + \frac{b-a}{c})^2$

$= 2c^2 + \frac{2c^3 A}{b-a} - \frac{c^4 A^2}{(b-a)^2} - \frac{2c^3 A}{b-a} - c^2$

$= c^2 - \frac{c^4 A^2}{(b-a)^2}$

Then, as $VarZ > 0$ (Z is not constant, so $VarZ \neq 0$),

$1 - \frac{c^2 A^2}{(b-a)^2} > 0$, so that $\frac{cA}{b-a} < 1$,

and hence $\ln\left|\frac{b-c}{a-c}\right| < \frac{b-a}{c}$ (as $b - a > 0$)

Also, $\frac{b-c}{a-c} = \frac{(b-c)/c}{(a-c)/c} = \frac{b/a}{a/b} > 0$,

so that $\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}$, as required.