STEP 2020, P3, Q10 - Solution (5 pages; 24/5/23)

1st Part

By N2L and Hooke's Law, $mg - \frac{(kmg)x}{a} = m\ddot{x}$,

where x is the extension of the spring.

Thus $\ddot{x} + \frac{kg}{a}x = g$ is the equation of motion of the particle. (*)

[In the Mark Scheme, x is the extension below the equilibrium,

and this produces the simpler equation $\ddot{x} = -\frac{kg}{a}x$]

2nd Part

The auxiliary eq'n is $\mu^2 + \frac{kg}{a} = 0$,

so that $\mu = i\sqrt{\frac{kg}{a}}$, and the Complementary Function is therefore
$x = Acos\left(\sqrt{\frac{kg}{a}}\, t\right) + Bsin\left(\sqrt{\frac{kg}{a}}\, t\right)$ [standard result]

Let $x = C$ (a constant) be the trial function for the Particular Integral.

Then, substituting into (*),

$\frac{kg}{a}C = g$, so that $C = \frac{a}{k}$,

and hence the general solution is

$x = Acos\left(\sqrt{\frac{kg}{a}}\ t\right) + Bsin\left(\sqrt{\frac{kg}{a}}\ t\right) + \frac{a}{k}$

When $t = 0, x = a$

Thus $a = A + \frac{a}{k}$, so that $A = a(1 - \frac{1}{k})$

As $\dot{x} = -\sqrt{\frac{kg}{a}}Asin\left(\sqrt{\frac{kg}{a}}\ t\right) + \sqrt{\frac{kg}{a}}Bcos\left(\sqrt{\frac{kg}{a}}\ t\right)$

When $t = 0,\ \dot{x} = 0,$

and so $0 = \sqrt{\frac{kg}{a}}B$, and so $B = 0$

So the particular solution is

$x = a(1 - \frac{1}{k})cos\left(\sqrt{\frac{kg}{a}}\ t\right) + \frac{a}{k}$

Thus the particle oscillates vertically [this could have been established from the Complementary Function alone].

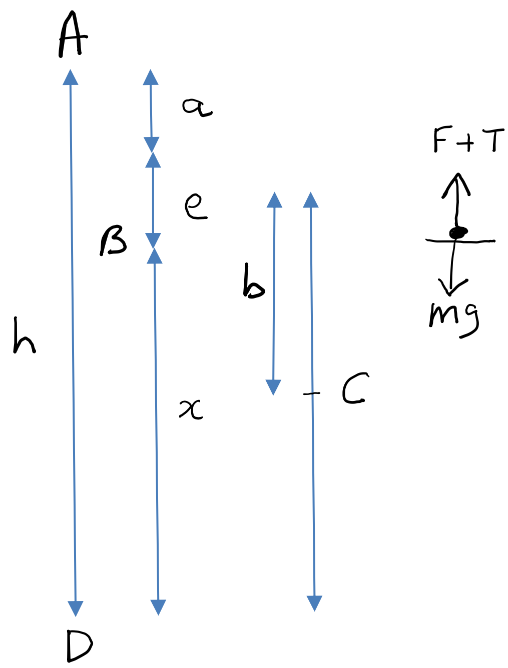
3rd Part

The period T satisfies $\sqrt{\frac{kg}{a}}\ T = 2\pi$, so that $T = \frac{2\pi}{\sqrt{\frac{kg}{a}}}$

Then $\frac{2\pi}{\sqrt{\frac{kg}{a}}} = \Omega \Rightarrow \frac{kg}{a} = \Omega^2$, and so $kg = a\Omega^2$, as required.

[The Mark Scheme just quotes $\Omega^2 = \frac{kg}{a}$ as a standard result from the differential equation.]

4th Part



Referring to the diagram above, the platform oscillates about C, with its lowest point being at D.

The forces on the particle are F (say), upwards from the platform, T upwards due to the tension in the spring, and the weight of the particle, mg.

The particle has the same acceleration $\ddot{x}$ (upwards) as the platform.

As the platform oscillates about C, $x = b + bsin(\omega t)$, where x is

the distance of the particle above D.

and so $\dot{x} = b\omega cos(\omega t)$, and $\ddot{x} = -b\omega^2 sin(\omega t) = -\omega^2(x - b)$

Then, applying N2L to the particle:

$F + T - mg = m\ddot{x}$

$\Rightarrow F = -m\omega^2(x-b) - \frac{kmge}{a} + mg$,

where e is the extension of the spring

(noting that e can be negative (with $T < 0$), as AB is a spring, rather than a string)

$= mg - m(h-a-x)\Omega^2 + m\omega^2(b-x)$,

as $kg = a\Omega^2$ and $h = a + e + x$

So $F = mg + m\Omega^2(a+x-h) + m\omega^2(b-x)$, as required.

5th Part

If the particle remains in contact with the platform, then $F \geq 0$,

ie $mg + m\Omega^2(a+x-h) + m\omega^2(b-x) \geq 0$,

so that $h\Omega^2 \leq g + \Omega^2(a+x) + \omega^2(b-x)$

$= \frac{a\Omega^2}{k} + \Omega^2(a+x) + \omega^2(b-x)$,

and hence $h \leq \frac{a}{k} + (a+x) + \frac{\omega^2}{\Omega^2}(b-x)$

$= a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + x(1-\frac{\omega^2}{\Omega^2})$ (*)

Then, $\omega < \Omega \Rightarrow 1 - \frac{\omega^2}{\Omega^2} > 0$. Also $x \geq 0$

So the smallest upper bound for h (when $x = 0$) is $a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$

ie $h \leq a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$, as required.

6th Part

When $\omega > \Omega$, $1 - \frac{\omega^2}{\Omega^2} < 0$. Also $x \leq 2b$.

So the smallest upper bound for h (when $x = 2b$) is

$$a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + 2b\left(1 - \frac{\omega^2}{\Omega^2}\right) = a\left(1 + \frac{1}{k}\right) + 2b - \frac{\omega^2 b}{\Omega^2} \;;$$

ie $h \leq a\left(1 + \frac{1}{k}\right) + 2b - \frac{\omega^2 b}{\Omega^2}$

7th Part

When $\omega = \Omega$, $h \leq a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + x(1 - \frac{\omega^2}{\Omega^2})$ (from (*)),

so that $h \leq a\left(1 + \frac{1}{k}\right) + b$

Writing $\lambda = \frac{\omega^2}{\Omega^2}$, the upper bound for h, $H(\lambda)$ say, satisfies:

$H(\lambda) = a\left(1 + \frac{1}{k}\right) + \lambda b$ when $\omega < \Omega$; ie when $\lambda < 1$,

$H(\lambda) = a\left(1 + \frac{1}{k}\right) + b$ when $\omega = \Omega$; ie when $\lambda = 1$,

and $H(\lambda) = a\left(1 + \frac{1}{k}\right) + 2b - \lambda b$ when $\omega > \Omega$; ie when $\lambda > 1$

Hence, for all values of ω, the upper bound for h doesn't exceed $a\left(1 + \frac{1}{k}\right) + b$, and so $h \leq a\left(1 + \frac{1}{k}\right) + b$

[Of course, when $\omega \neq \Omega$ a smaller upper bound for h applies – dependent on ω.]