STEP 2019, P3, Q5 - Solution (3 pages; 21/7/20)

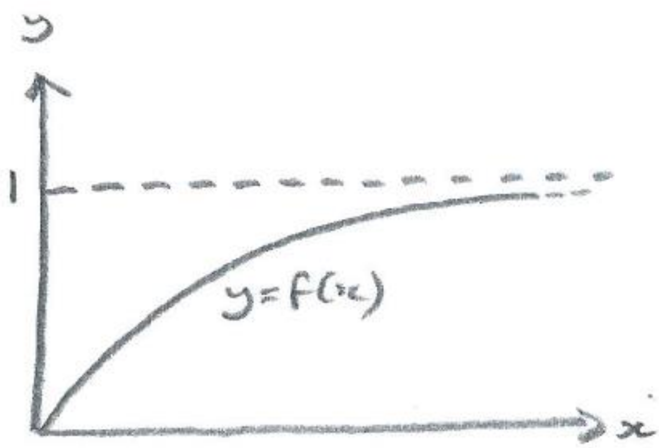
[Note the Erratum at the start of the paper.]

(i) $f(0) = 0$ and $f(x) \to 1^-$ as $x \to \infty$

$f(x) = \frac{x}{\sqrt{x^2+p}} \Rightarrow f'(x) = \frac{\sqrt{x^2+p}-x.\frac{1}{2}(x^2+p)^{-\frac{1}{2}}.2x}{x^2+p}$

$= \frac{(x^2+p)-x^2}{(x^2+p)^{\frac{3}{2}}} = \frac{p}{(x^2+p)^{\frac{3}{2}}} > 0$ for $x \geq 0$

And $f''(x) = p\left(-\frac{3}{2}\right)(x^2+p)^{-\frac{5}{2}}(2x) < 0$ for $x > 0$



(ii) **1st part**

If $y = \frac{cx}{\sqrt{x^2+p}}$, then $\frac{dy}{dx} = \frac{cp}{(x^2+p)^{\frac{3}{2}}}$, from (i)

And $y^2 = \frac{c^2x^2}{x^2+p}$, so that $b^2 - y^2 = \frac{b^2(x^2+p)-c^2x^2}{x^2+p}$

and $c^2 - y^2 = \frac{c^2(x^2+p)-c^2x^2}{x^2+p} = \frac{c^2p}{x^2+p}$

Then $I = \int \frac{1}{\left(\frac{b^2(x^2+p)-c^2x^2}{x^2+p}\right)\sqrt{\frac{c^2p}{x^2+p}}} \frac{cp}{(x^2+p)^{\frac{3}{2}}}\, dx$

$= \int \frac{\sqrt{p}}{b^2(x^2+p)-c^2x^2}\, dx$

And if $p = 1, I = \int \frac{1}{b^2+(b^2-c^2)x^2}\ dx$, as required.

2nd part

Let $b = \sqrt{3}$ & $c = \sqrt{2}$

Then, with the substitution $y = \frac{x\sqrt{2}}{\sqrt{x^2+1}}$,

$y = 1 \Rightarrow x^2 + 1 = 2x^2$, so that $x = 1$ (rejecting the spurious sol'n $x = -1$, as it doesn't give the correct value for y)

and $y = \sqrt{2} \Rightarrow x^2 + 1 = x^2$, so that $x = \infty$,

and $J = \int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}} dy$ becomes $\int_1^{\infty} \frac{1}{3+(3-2)x^2}\ dx$, from the 1st part

Hence $J = \frac{1}{\sqrt{3}}\left[tan^{-1}\frac{x}{\sqrt{3}}\right]_1^{\infty} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

3rd part

Let $z = \frac{1}{y}$, so that $dz = -\frac{1}{y^2} dy$

[noting that the limits of integration will become the limits of the previous integral reversed]

Then $\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}} dy = \int_{\sqrt{2}}^{1} \frac{\left(\frac{1}{z}\right)}{(\frac{3}{z^2}-1)\sqrt{\frac{2}{z^2}-1}} \left(-\frac{1}{z^2}\right) dz$

$= \int_1^{\sqrt{2}} \frac{1}{(3-z^2)\sqrt{2-z^2}}\ dz = \frac{\pi}{3\sqrt{3}}$, from the 2nd part.

(iii) We could try a substitution of the form $y = \frac{cx}{\sqrt{x^2+p}}$

Then $y^2 = \frac{c^2x^2}{x^2+p}$ and $2y^2 - 1 = \frac{2c^2x^2-(x^2+p)}{x^2+p} = \frac{(2c^2-1)x^2-p}{x^2+p}$

The corresponding expression in the 1st part was $\frac{c^2p}{x^2+p}$, and this had to be positive, in order for the square root to be defined.

So we could try setting $c = \frac{1}{\sqrt{2}}$ & $p = -1$

Then $3y^2 - 1 = \frac{\frac{3}{2}x^2-(x^2-1)}{x^2-1} = \frac{\frac{1}{2}x^2+1}{x^2-1}$

When $y = \frac{1}{\sqrt{2}}, \frac{1}{2} = \frac{\frac{1}{2}x^2}{x^2-1} \Rightarrow x^2 - 1 = x^2 \Rightarrow x = \infty$

When $y = 1$, $1 = \frac{\frac{1}{2}x^2}{x^2-1} \Rightarrow x^2 - 1 = \frac{1}{2}x^2 \Rightarrow x = \sqrt{2}$ ($x = -\sqrt{2}$ is rejected, as it gives $y = -1$)

From (i), $\frac{dy}{dx} = \frac{-\left(\frac{1}{\sqrt{2}}\right)}{(x^2-1)^{\frac{3}{2}}}$, so that $\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\, dy$

$$= \int_{\infty}^{\sqrt{2}} \frac{1}{\left(\frac{\frac{1}{2}x^2+1}{x^2-1}\right)\sqrt{\frac{1}{x^2-1}}} \frac{-\left(\frac{1}{\sqrt{2}}\right)}{(x^2-1)^{\frac{3}{2}}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int_{\sqrt{2}}^{\infty} \frac{1}{\left(\frac{1}{2}x^2+1\right)}\, dx$$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}} tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$