STEP 2019, P3, Q12 - Solution (3 pages; 23/7/20)

initial part

Considering each of the integers 1 to n in turn, either it is included in a particular subset, or it isn't; ie there are 2 choices, made n times; giving 2^n possibilities.

(i) $\frac{1}{2}$: For a given subset, the integer 1 is equally likely to be included or excluded. [Alternatively, $\frac{no.\ of\ subsets\ that\ include\ 1}{total\ no.\ of\ subsets}$

$= \frac{total\ no.\ of\ subsets\ formed\ from\ the\ integers\ 1\ to\ n-1\ (to\ go\ with\ 1)}{2^n}$

$= \frac{2^{n-1}}{2^n} = \frac{1}{2}$]

(ii) **1st part**

$\mathrm{P}(A_1 \cap A_2 = \emptyset)$

$= P(A_1 \& A_2\ don't\ both\ contain\ the\ integer\ 1)$

$\times P(A_1 \& A_2\ don't\ both\ contain\ 2\ |\ they\ don't\ both\ contain\ 1)$

$\times P(A_1 \& A_2\ don't\ both\ contain\ 3\ |\ they\ don't\ both\ contain\ 1\ \&\ 2) \ldots$

$= P(A_1 \& A_2\ don't\ both\ contain\ 1)$

$\times P(A_1 \& A_2\ don't\ both\ contain\ 2)$

$\times P(A_1 \& A_2\ don't\ both\ contain\ 3) \ldots$

(the number of subsets containing the integer 2 and the integer 1 equals the number of subsets containing the integer 2, but not containing the integer 1)

$= [P(A_1\ \&\ A_2\ don't\ both\ contain\ 1)]^n$

$= [1 - P(A_1 doesn't\ contain\ 1) \times P(A_2\ doesn't\ contain\ 1)]^n$

(as A_1 & A_2 are independent)

$= [1 - P(1 \in A_1)^2]^n$

$= (1 - \frac{1}{4})^n = \left(\frac{3}{4}\right)^n$, as required.

2nd part

$\text{P}(A_1 \cap A_2 \cap A_3 = \emptyset) = [1 - P(1 \in A_1)^3]^n$

$= (1 - \frac{1}{8})^n = \left(\frac{7}{8}\right)^n$

3rd part

$\text{P}(A_1 \cap A_2 \cap \ldots \cap A_m = \emptyset) = [1 - P(1 \in A_1)^m]^n$

$= (1 - \frac{1}{2^m})^n$

(iii) **1st part**

The following table shows which events are consistent with $A_1 \subseteq A_2$:

	$1 \in A_1$	$1 \notin A_1$
$1 \in A_2$	Yes	Yes
$1 \notin A_2$	No	Yes

So $P(A_1 \subseteq A_2) = [1 - P(1 \in A_1)P(1 \notin A_2)]^n$

$= \left[1 - \frac{1}{2} . \frac{1}{2}\right]^n$

$= \left(\frac{3}{4}\right)^n$

2nd part

The events that **are** consistent with $A_1 \subseteq A_2 \subseteq A_3$ are:

$1 \in A_1$ and $1 \in A_2$ and $1 \in A_3$

$1 \notin A_1$ and $1 \in A_2$ and $1 \in A_3$

$1 \notin A_1$ and $1 \notin A_2$ and $1 \in A_3$

$1 \notin A_1$ and $1 \notin A_2$ and $1 \notin A_3$

[It is possible to list the events that **aren't** consistent with

$A_1 \subseteq A_2 \subseteq A_3$ (ie the method of the 1st part), but it isn't as easy to extend the method to the case involving A_m.]

These events are mutually exclusive and each has probability

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

So $P(A_1 \subseteq A_2 \subseteq A_3) = \left[\frac{4}{8}\right]^n = \frac{1}{2^n}$

3rd part

Extending the 2nd part, there are $m + 1$ mutually exclusive events that **are** consistent with $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_m$, and each of these has

probability $\left(\frac{1}{2}\right)^m$

So $P(A_1 \subseteq A_2 \subseteq \ldots \subseteq A_m) = \left[\frac{m+1}{2^m}\right]^n$ or $\frac{(m+1)^n}{2^{mn}}$