STEP 2019, P3, Q11 - Solution (3 pages; 5/2/21)

(i) $P(r$ customers take sand)

$= \sum_{k=r}^{\infty} P(k \text{ customers})P(r \text{ take sand}|k \text{ customers})$

$= \sum_{k=r}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\binom{k}{r}p^r(1-p)^{k-r}$

$= e^{-\lambda}p^r\sum_{k=r}^{\infty}\frac{\lambda^k}{k!}\frac{k!}{r!(k-r)!}(1-p)^{k-r}$

writing $i = k - r$

$= \frac{e^{-\lambda}p^r\lambda^r}{r!}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}(1-p)^i$

$= \frac{e^{-\lambda}p^r\lambda^r}{r!}e^{\lambda(1-p)}$

$= \frac{e^{-p\lambda}(p\lambda)^r}{r!}$,

which is the Poisson probability for a variable with a mean of $p\lambda$

Also, the conditions for a Poisson variable are met (given that the number of customers meets these conditions):

- events are random and independent, and are rare and occur singly

- constant parameter $p\lambda$

(ii)

customer	takes	left
1	kS	$S - kS = (1-k)S$
2	$k(1-k)S$	$(1-k)S - k(1-k)S = (1-k)^2S$
3	$k(1-k)^2S$	$(1-k)^2S - k(1-k)^2S = (1-k)^3S$

The total amount taken if there are r customers who take the sand is $kS + k(1-k)S + k(1-k)^2S + \cdots + k(1-k)^{r-1}S$

$= \frac{kS[1-(1-k)^r]}{1-(1-k)} = S[1-(1-k)^r]$

(and note that this is valid for $r = 0$)

[but as the denominator is k, this only applies if $k \neq 0$]

Then the expected total amount taken is

$\sum_{r=0}^{\infty} e^{-p\lambda} \frac{(p\lambda)^r}{r!} S[1-(1-k)^r]$

$= S.1 - S\sum_{r=0}^{\infty} e^{-p\lambda} \frac{(p\lambda[1-k])^r}{r!}$

$= S - Se^{-kp\lambda} \sum_{r=0}^{\infty} e^{-p\lambda+kp\lambda} \frac{(p\lambda[1-k])^r}{r!}$

$= S - Se^{-kp\lambda} \sum_{r=0}^{\infty} e^{-p\lambda(1-k)} \frac{(p\lambda[1-k])^r}{r!}$

$= S - Se^{-kp\lambda}.1$

$= \left(1 - e^{-kp\lambda}\right)S$, as required

[If $k = 0$, then no sand is taken, and the formula gives the correct value.]

(iii) **1st part**

If there are r customers during the day who take the free sand, then (from the table in (ii)) the amount of sand left, before the assistant takes his share, is $(1-k)^r S$.

Then Prob(assistant takes golden grain) $= \frac{k(1-k)^r S}{S} = k(1-k)^r$

So required prob.

$= \sum_{r=0}^{\infty} e^{-p\lambda} \frac{(p\lambda)^r}{r!} .k(1-k)^r$

$= ke^{-kp\lambda} \sum_{r=0}^{\infty} e^{-p\lambda(1-k)} \frac{(p\lambda[1-k])^r}{r!}$

$= ke^{-kp\lambda}$

2nd part

If $k = 0$, no one takes any sand (including the assistant), and the required prob. is zero. This agrees with the formula $ke^{-kp\lambda}$, and $k = 0$ doesn't invalidate the derivation of the formula.

3rd part

As $k \to 1$, it becomes increasingly likely that one of the customers will have taken the golden grain, and the required prob. $ke^{-kp\lambda}$ tends to the probability that no customers take any sand; ie $e^{-p\lambda}$

4th part

Let $p(k) = ke^{-kp\lambda}$

Then $p'(k) = e^{-kp\lambda} - kp\lambda e^{-kp\lambda}$

and $p'(k) = 0$ when $1 - kp\lambda = 0$;

ie when $k = \frac{1}{p\lambda}$ (as $p\lambda > 1, \frac{1}{p\lambda} < 1$)

To confirm that $p(k)$ is maximised at this value,

$p''(k) = -p\lambda e^{-kp\lambda} - p\lambda p'(k)$

$= -p\lambda e^{-kp\lambda} - p\lambda(e^{-kp\lambda} - kp\lambda e^{-kp\lambda})$

$= e^{-kp\lambda}(-2p\lambda + (p\lambda)^2 k)$

and $p''\left(\frac{1}{p\lambda}\right) = e^{-1}(-2p\lambda + p\lambda) = -p\lambda e^{-1} < 0$

[There are several 'typos' in the official Hints and Mark scheme, where a k is missing from the amount that the merchant's assistant takes.]