STEP 2019, P2, Q7 - Solution (4 pages; 23/3/22)

(i) **1st part**

$\underline{a} + \underline{b} + \underline{c} = \underline{0}$

Taking the scalar product of both sides, with $\underline{a}$, $\underline{b}$ & $\underline{c}$ in turn,

$1 + \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} = 0$

$\underline{a} \cdot \underline{b} + 1 + \underline{b} \cdot \underline{c} = 0$

$\underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c} + 1 = 0$

Writing $x = \underline{a} \cdot \underline{b}$, $y = \underline{a} \cdot \underline{c}$ & $z = \underline{b} \cdot \underline{c}$,

$1 + x + y = 0$

$x + 1 + z = 0$

$y + z + 1 = 0$

Substituting for z from the 3rd eq'n into the 2nd,

$1 + x + y = 0$

$x + 1 + (-y - 1) = 0; x = y$

Hence $1 + 2x = 0$, and $\underline{a} \cdot \underline{b} = x = -\frac{1}{2}$

2nd part

[Due to the symmetry between $\underline{a}$, $\underline{b}$ & $\underline{c}$, the answer is bound to be that it's an equilateral triangle, but obviously this has to be proved.]

[The official 'Hints & Sol'ns' just accepts the fact that $\underline{a} \cdot \underline{b} = -\frac{1}{2} \Rightarrow$ the angle between $\underline{a}$ & $\underline{b}$ is 120°, as $|\underline{a}| = |\underline{b}| = 1$, together with a (3d) sketch, invoking symmetry presumably.]

Consider the angle between sides AB and AC (θ, say).

Then $\overrightarrow{AB}.\overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|cos\theta$ (1)

so that $(\underline{b} - \underline{a}).(\underline{c} - \underline{a}) = |\underline{b} - \underline{a}||\underline{c} - \underline{a}|cos\theta$

LHS of (1) $= \underline{b} \,.\, \underline{c} - \underline{a} \,.\, \underline{b} - \underline{a} \,.\, \underline{c} + 1$

By symmetry, $\underline{b} \,.\, \underline{c} = \underline{a} \,.\, \underline{c} = \underline{a} \,.\, \underline{b} = -\frac{1}{2}$,

so that LHS $= \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) + 1 = \frac{3}{2}$

For the RHS of (1):

$|\underline{b} - \underline{a}|^2 = (\underline{b} - \underline{a}).(\underline{b} - \underline{a}) = 1 - 2\underline{a} \,.\, \underline{b} + 1 = 2 - 2\left(-\frac{1}{2}\right) = 3$

and by symmetry $|\underline{c} - \underline{a}|^2 = |\underline{c} - \underline{b}|^2 = 3$ also,

so that all the sides are equal, and the triangle ABC is equilateral

[Also, (1) gives $\frac{3}{2} = \sqrt{3}.\sqrt{3}cos\theta$, so that $cos\theta = \frac{1}{2}$; $\theta = 60°$, and hence, by symmetry, all 3 angles are $60°$.]

(ii) **1st part**

$\underline{a}_1 + \underline{a}_2 + \underline{a}_3 + \underline{a}_4 = \underline{0}$

Taking the scalar product of both sides with $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ & $\underline{a}_4$, in turn, and writing $\underline{a}_1.\underline{a}_3 = x$, $\underline{a}_1.\underline{a}_4 = y$, $\underline{a}_2.\underline{a}_3 = z$, $\underline{a}_2.\underline{a}_4 = w$:

$1 + \underline{a}_1.\underline{a}_2 + x + y = 0$ (1)

$\underline{a}_1.\underline{a}_2 + 1 + z + w = 0$ (2)

$x + z + 1 + \underline{a}_3.\underline{a}_4 = 0$ (3)

$y + w + \underline{a}_3.\underline{a}_4 + 1 = 0$ (4)

From (1) & (2), $x + y = z + w$ (5)

From (3) & (4), $x + z = y + w$ (6)

Subtracting (6) from (5): $y - z = z - y \Rightarrow 2y = 2z \Rightarrow y = z$

Then (5) $\Rightarrow x = w$, and (1) - (4) become:

$1 + \underline{a}_1.\underline{a}_2 + x + y = 0$ (1)

$x + y + 1 + \underline{a}_3.\underline{a}_4 = 0$ (3'),

so that $\underline{a}_1.\underline{a}_2 = \underline{a}_3.\underline{a}_4$, as required.

(a) [Imagining the quadrilateral as suspended from a point (O) by 4 strings of unit length attached to its corners, a rectangle seems likely. Note that A_1 & A_2 (for example) are specified to be next to each other, so that there isn't symmetry between the 4 points, and a square is therefore not inevitable.]

From the working to the 1st part of (ii), $x = w$, so that

$x = \underline{a}_1.\underline{a}_3 = \underline{a}_2.\underline{a}_4$, and $y = z$, so that $y = \underline{a}_1.\underline{a}_4 = \underline{a}_2.\underline{a}_3$

Let $v = \underline{a}_1.\underline{a}_2 = \underline{a}_3.\underline{a}_4$

Consider the side A_1A_2: $\left|\overrightarrow{A_1A_2}\right|^2 = \overrightarrow{A_1A_2}.\overrightarrow{A_1A_2}$

$= (\underline{a}_2 - \underline{a}_1).(\underline{a}_2 - \underline{a}_1) = 1 - 2\underline{a}_1.\underline{a}_2 + 1 = 2(1 - v)$

Similarly, $\left|\overrightarrow{A_3A_4}\right|^2 = 2(1 - \underline{a}_3.\underline{a}_4) = 2(1 - v)$,

so that $A_1A_2 = A_3A_4$

Also $\left|\overrightarrow{A_1A_4}\right|^2 = 1 - 2\underline{a}_1.\underline{a}_4 + 1 = 2(1 - y)$

and $|\overrightarrow{A_2A_3}|^2 = 1 - 2\underline{a}_2.\underline{a}_3 + 1 = 2(1-y)$,

so that $A_1A_4 = A_2A_3$

So far, we have established that $A_1A_2A_3A_4$ is a parallelogram.

Now consider the diagonals A_1A_3 & A_2A_4:

$|\overrightarrow{A_1A_3}|^2 = 1 - 2\underline{a}_1.\underline{a}_3 + 1 = 2(1-x)$

and $|\overrightarrow{A_2A_4}|^2 = 1 - 2\underline{a}_2.\underline{a}_4 + 1 = 2(1-x)$,

so that $A_1A_3 = A_2A_4$, and hence $A_1A_3\ A_2A_4$ is a rectangle.

[The official mark scheme doesn't offer any explanation as to why the shape should be a rectangle.]

(b) As the tetrahedron is regular,

$A_1A_2 = A_1A_3 = A_1A_4$,

so that $|\overrightarrow{A_1A_2}|^2 = |\overrightarrow{A_1A_3}|^2 = |\overrightarrow{A_1A_4}|^2$,

and so $2(1-v) = 2(1-x) = 2(1-y)$, from the working for (a).

Thus $x = y = v$.

Then, as $1 + v + x + y = 0$, from (1) in the 1st part of (ii),

$x = -\frac{1}{3}$, and the sides of the tetrahedron are

$A_1A_2 = \sqrt{2\left(1-[-\frac{1}{3}]\right)} = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$