# **STEP 2019, P2, Q1 - Solution** (3 pages; 10/7/20)

**1st part**

$f'(x) = g(x) + (x - p)g'(x)$

$f'(a) = g(a) + (a - p)g'(a)$

The tangent to the curve $y = f(x)$ at $x = a$ is

$y - f(a) = [g(a) + (a - p)g'(a)](x - a)$

When this tangent passes through $(p, 0)$,

$-f(a) = [g(a) + (a - p)g'(a)](p - a) \quad (1)$

And $f(a) = (a - p)g(a)$,

so that $(1) \Leftrightarrow (p - a)g(a) = [g(a) + (a - p)g'(a)](p - a)$

$\Leftrightarrow 0 = (a - p)g'(a)$

$\Leftrightarrow g'(a) = 0$, as $a \neq p$ , as required.

(i) **1st part**

From the initial result, $g'(a) = 0$, where

$g(x) = A(x - q)(x - r)$,

so that $g'(x) = A(x - r) + A(x - q)$

and $g'(a) = 0 \Rightarrow a - r + a - q = 0$ (as $A \neq 0$);

ie $2a = q + r$, as required. (1)

**2nd part**

Writing $f(x) = A(x - p)(x - q)(x - r)$,

the eq'n of the tangent is $y - f(a) = f'(a)(x - a)$

and, as the tangent passes through $(p, 0)$,

$-f(a) = f'(a)(p-a)$

Also $f(a) = A(a-p)(a-q)(a-r)$,

so that $f'(a) = \frac{-A(a-p)(a-q)(a-r)}{p-a} = A(a-q)(a-r)$

Then, from (1), the gradient of the tangent at $x = a$,

$f'(a) = A\left(\frac{q+r}{2} - q\right)\left(\frac{q+r}{2} - r\right) = \frac{A}{4}(r-q)(q-r)$

$= -\frac{A}{4}(r-q)^2$

(ii) As before, but with $r$ in place of $p$, $2c = p + q$

and $f'(c) = -\frac{A}{4}(q-p)^2$

So the tangent at $x = c$ is parallel to the tangent at $x = a$ if and only if $-\frac{A}{4}(q-p)^2 = -\frac{A}{4}(r-q)^2 \;\Leftrightarrow\; q - p = r - q$

(as $q - p$ & $r - q$ are both positive)

$\Leftrightarrow 2q = p + r \quad (2)$

Now, $f(x) = A(x-p)(x-q)(x-r)$

$\Rightarrow f'(x) = A(x-q)(x-r) + A(x-p)(x-r) + A(x-p)(x-q)$

so that $f'(q) = A(q-p)(q-r)$

and the eq'n of the tangent at $x = q$ is

$y - 0 = A(q-p)(q-r)(x-q)$

The tangent at $x = q$ meets the curve when

$A(q-p)(q-r)(x-q) = A(x-p)(x-q)(x-r)$

$\Leftrightarrow x = q$ or $(q-p)(q-r) = (x-p)(x-r)$

$\Leftrightarrow x = q$ or $x^2 - (p+r)x - q^2 + q(p+r) = 0$   (3)

When (2) is satisfied, so that $2q = p + r$,

(3) becomes $x = q$ or $x^2 - 2qx - q^2 + 2q^2 = 0$

$\Leftrightarrow x = q$ or $(x-q)^2 = 0$

ie the tangent at $x = q$ only meets the curve at $x = q$, and so does not meet the curve again.

Conversely, if the tangent at $x = q$ does not meet the curve again, the only roots of (3) will be $x = q$

(3) $\Rightarrow x = \frac{p+r\pm\sqrt{(p+r)^2-4[-q^2+q(p+r)]}}{2}$

Hence $(p+r)^2 - 4[-q^2 + q(p+r)] = 0$ and $q = \frac{p+r}{2}$

If $q = \frac{p+r}{2}$, the LHS of the 1st equation is $4q^2 + 4q^2 - 4q(2q) = 0$

So, if the tangent at $x = q$ does not meet the curve again, (2) is satisfied, and the tangent at $x = c$ is parallel to the tangent

at $x = a$

Thus, the tangent at $x = c$ is parallel to the tangent at $x = a$ if and only if the tangent at $x = q$ does not meet the curve again, as required.