STEP 2019, P2, Q11 - Solution (5 pages;7/7/20)

(i) **1st part**

When $n_3 = 9$, possibilities are:

2, 8

3, 8 ; 3, 7

4, 8; 4,7; 4, 6

5, 8; 5, 7; 5, 6

6, 8; 6, 7

7, 8

ie number of ways is 12

When $n_3 = 10$, possibilities are:

2, 9

3, 9 ; 3, 8

4, 9; 4,8; 4, 7

5, 9; 5, 8; 5, 7; 5, 6

6, 9; 6, 8; 6, 7

7, 9; 7, 8

8, 9

ie number of ways is 16

2nd part

For $n_3 = 7$, we have:

2, 6

3, 6; 3, 5

$4, 6; 4, 5$

$5, 6$

So, for $n_3 = 2n + 1$, the numbers of ways are:

1

2

...

$n - 1$

$n - 1$

...

2

1

giving $2(1 + 2 + \cdots + [n - 1]) = (n - 1)n$

3rd part

For $n_3 = 2n$, the numbers of ways are:

$1 + 2 + \cdots + (n - 2) + (n - 1) + (n - 2) + \cdots + 2 + 1$

$= 2(1 + 2 + \cdots + [n - 2]) + (n - 1)$

$= (n - 2)(n - 1) + (n - 1) = (n - 1)(n - 1) = (n - 1)^2$

(ii) When $N = 2n + 1$, Prob. that triangle can be formed

$= \frac{number\ of\ ways\ in\ which\ a\ triangle\ can\ be\ formed}{number\ of\ ways\ in\ which\ the\ rods\ can\ be\ selected}$

$= \frac{(n-1)n}{\binom{2n}{2}} = \frac{(n-1)n}{\left(\frac{2n(2n-1)}{2}\right)} = \frac{n-1}{2n-1}$, as required.

When $N = 2n$, Prob. $= \frac{(n-1)^2}{\binom{2n-1}{2}} = \frac{(n-1)^2}{(\frac{(2n-1)(2n-2)}{2})} = \frac{n-1}{2n-1}$ also.

(iii) Prob. that triangle can be formed

$= \sum_{r=4}^{2M+1}\{Prob(longest\ rod\ is\ of\ length\ r)$

$\times\, Prob(\text{triangle can be formed}|longest\ rod\ is\ of\ length\ r)\}$

$Prob(longest\ rod\ is\ of\ length\ r)$

$= \frac{no.\ of\ ways\ in\ which\ longest\ rod\ is\ of\ length\ r}{no.\ of\ ways\ in\ which\ 3\ rods\ can\ be\ selected} = \frac{\binom{r-1}{2}}{\binom{2M+1}{3}}$

$= \frac{\left(\frac{(r-1)(r-2)}{2!}\right)}{\left(\frac{(2M+1)(2M)(2M-1)}{3!}\right)} = \frac{3(r-1)(r-2)}{2M(2M+1)(2M-1)}$

and P$rob(\text{triangle can be formed}|longest\ rod\ is\ of\ length\ r)$

$= \frac{n-1}{2n-1}$ when $r = 2n+1$, and $\frac{n-1}{2n-1}$ when $r = 2n$

So Prob. that triangle can be formed

$= \sum_{n=2}^{M} \frac{3(2n-1)(2n-2)}{2M(2M+1)(2M-1)} \cdot \frac{n-1}{2n-1}$

$+ \sum_{n=2}^{M} \frac{3(2n)(2n-1)}{2M(2M+1)(2M-1)} \cdot \frac{n-1}{2n-1}$

$= \frac{3}{M(2M+1)(2M-1)} \sum_{n=2}^{M}\{(n-1)^2 + n(n-1)\}$

$= \frac{3}{M(2M+1)(2M-1)} \sum_{n=1}^{M}(n-1)(2n-1)$

Now $(n-1)(2n-1) = 2n^2 - 3n + 1$

and $\sum_{n=1}^{M}\{2n^2 - 3n + 1\}$

$= \frac{2}{6}M(M+1)(2M+1) - \frac{3}{2}M(M+1) + M$

$= \frac{M}{6}\{4M^2 + 6M + 2 - 9M - 9 + 6\}$

$= \frac{M}{6}\{4M^2 - 3M - 1\}$

$= \frac{M(4M+1)(M-1)}{6}$

Then Prob. that triangle can be formed

$= \frac{3}{M(2M+1)(2M-1)} \cdot \frac{M(4M+1)(M-1)}{6}$

$= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$, as required.

Alternative method

Prob. that triangle can be formed

$= \frac{no.\ of\ ways\ in\ which\ a\ triangle\ can\ be\ formed}{no.\ of\ ways\ in\ which\ 3\ rods\ can\ be\ selected}$

From (i), no. of ways in which a triangle can be formed

$= \sum_{k=4}^{2M+1}$(no. of ways in which a triangle can be formed

from the integers 1,2, … k)

$= \sum_{r=2}^{M}(r-1)^2 + \sum_{r=2}^{M}(r-1)r$

(for $k = 2r$ & $k = 2r + 1$, respectively)

$= \sum_{r=2}^{M}(r-1)(2r-1)$

$= \sum_{r=1}^{M}(r-1)(2r-1)$

$= \frac{M(4M+1)(M-1)}{6}$, as above

Hence Prob. that triangle can be formed

$$= \frac{\left(\frac{M(4M+1)(M-1)}{6}\right)}{\binom{2M+1}{3}}$$

$$= \frac{\left(\frac{M(4M+1)(M-1)}{6}\right)}{\left(\frac{(2M+1)(2M)(2M-1)}{3!}\right)}$$

$$= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$$