STEP 2019, P1, Q4 - Solution (3 pages; 12/2/21)

(i) $\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}$

$\Rightarrow 3 + 2\sqrt{2} = m^2 + 2n^2 + 2mn\sqrt{2}$

As we are only being asked to find a pair of integers that work,

we can see that $m = n = 1$ is a solution.

[Algebraically:

Beware of spurious sol'ns, arising from $-\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}$.

We require $mn = 1$,

and $3 = m^2 + 2n^2 = \frac{1}{n^2} + 2n^2$

Let $N = n^2$, so that $2N^2 - 3N + 1 = 0$,

and $(2N-1)(N-1) = 0$,

so that $N = 1$ (in order for $n = \pm 1$; ie an integer)

If $n = 1$, then $m = 1$, and $m + n\sqrt{2} > 0$, and therefore this isn't the spurious sol'n from $-\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}$]

(ii) **1st Part**

[Note that it isn't sufficient to expand the given form, and observe that the coefficient of x^3 is zero, as this is proving that $B \Rightarrow A$, rather than $A \Rightarrow B$.]

If $f(x) = 0$ has roots $\alpha, \beta, \gamma, \delta$,

then $f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= (x^2 - (\alpha+\beta)x + \alpha\beta)^2(x^2 - (\gamma+\delta)x + \gamma\delta)^2$

Also, as the coefficient of x^3 in $f(x)$ is zero, $\alpha+\beta+\gamma+\delta = 0$, so that if $-(\alpha+\beta) = s$, then $-(\gamma+\delta) = -s$

So, with $p = \alpha\beta$ and $q = \gamma\delta$, $f(x)$ can be written in the required form.

2nd Part

Expanding the given form for $f(x)$ gives

$x^4 + x^3(-s + s) + x^2(q - s^2 + p) + x(sq - ps) + pq,$

and $f(x)$ can therefore be written in the given form if the following equations can be solved:

$q - s^2 + p = -10 \ \ (1)$

$s(q - p) = 12 \ \ (2)$

$pq = -2 \ \ (3)$

3rd Part

Then $s^2(s^2 - 10)^2 + 8s^2 - 144$

$= \frac{144}{(q-p)^2}(q + p)^2 + \frac{8(144)}{(q-p)^2} - 144$, from (1) & (2)

$= \frac{144}{(q-p)^2}\{(q + p)^2 + 8 - (q - p)^2\}$

$= \frac{144}{(q-p)^2}\{8 + 4qp\} = 0,$ from (3)

4th Part

Writing $x = s^2$, $s^2(s^2 - 10)^2 + 8s^2 - 144 = 0$

$\Rightarrow g(x) = x(x - 10)^2 + 8x - 144 = 0$

Now $g(2) = 2(64) + 16 - 144 = 0,$

so that $g(x) = x^3 - 20x^2 + 108x - 144$

$= (x - 2)(x^2 - 18x + 72)$

$= (x - 2)(x - 6)(x - 12),$

so that the 3 possible values of s^2 are 2, 6 & 12

5th Part

With $s^2 = 2$, eq'ns (1) & (3) become:

$q + p = -8$ $(1')$ and $pq = -2$ (3)

so that $-\frac{2}{p} + p + 8 = 0$

$\Rightarrow p^2 + 8p - 2 = 0$

$\Rightarrow p = \frac{-8 \pm \sqrt{64+8}}{2} = -4 \pm 3\sqrt{2}$

Thus, $s = \sqrt{2}\,, p = -4 - 3\sqrt{2}\,, q = -4 + 3\sqrt{2}$ are possible values, as then eq'n (2): $s(q - p) = 12$ is satisfied.

So $f(x) = (x^2 + \sqrt{2}x - 4 - 3\sqrt{2})(x^2 - \sqrt{2}x - 4 + 3\sqrt{2})$

$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{2+16+12\sqrt{2}}}{2}$ or $\frac{\sqrt{2} \pm \sqrt{2+16-12\sqrt{2}}}{2}$

ie $\frac{-\sqrt{2} \pm \sqrt{6}\sqrt{3+2\sqrt{2}}}{2}$ or $\frac{\sqrt{2} \pm \sqrt{6}\sqrt{3-2\sqrt{2}}}{2}$ (*)

From (i), $\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}$

Similarly, for $\sqrt{3 - 2\sqrt{2}} = m + n\sqrt{2}$,

$3 - 2\sqrt{2} = m^2 + 2n^2 + 2mn\sqrt{2}$

We require $mn = -1$,

and $3 = m^2 + 2n^2 = \frac{1}{n^2} + 2n^2$ (as before)

and $\sqrt{3 - 2\sqrt{2}} = -1 + \sqrt{2}$ works

Then, from (*), $x = \frac{-\sqrt{2} \pm \sqrt{6}(1+\sqrt{2})}{2}$ or $\frac{\sqrt{2} \pm \sqrt{6}(-1+\sqrt{2})}{2}$

ie $\frac{-\sqrt{2}+\sqrt{6}+2\sqrt{3}}{2}$, $\frac{-\sqrt{2}-\sqrt{6}-2\sqrt{3}}{2}$, $\frac{\sqrt{2}-\sqrt{6}+2\sqrt{3}}{2}$ or $\frac{\sqrt{2}+\sqrt{6}-2\sqrt{3}}{2}$