STEP 2019, P1, Q3 - Solution (3 pages; 1/7/20)

1st part

$\int_0^{\frac{\pi}{4}} \frac{1}{1+sinx} dx = \int_0^{\frac{\pi}{4}} \frac{1-sinx}{1-sin^2x} dx$

$= \int_0^{\frac{\pi}{4}} \frac{1-sinx}{1-(1-cos^2x)} dx$

$= \int_0^{\frac{\pi}{4}} \frac{1-sinx}{cos^2x} dx$

$= \int_0^{\frac{\pi}{4}} sec^2x\, dx + \int_0^{\frac{\pi}{4}} \frac{-sinx}{cos^2x} dx \quad (1)$

Let $u = cosx$ in the 2nd integral.

Then $(1) = [tanx]_0^{\frac{\pi}{4}} + \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u^2}\, du$

$= 1 - 0 + \left[-\frac{1}{u}\right]_1^{\frac{1}{\sqrt{2}}}$

$= 1 - \sqrt{2} + 1 = 2 - \sqrt{2}$

[Alternatively, $\int_0^{\frac{\pi}{4}} \frac{-sinx}{cos^2x} dx = -\int_0^{\frac{\pi}{4}} tanxsecx\, dx = -[secx]_0^{\frac{\pi}{4}}$

$= -(\sqrt{2} - 1) = 1 - \sqrt{2}$]

2nd part

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{1+secx} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-secx}{1-sec^2x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-secx}{-tan^2x} dx$

$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cot^2x\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cosx}{sin^2x} dx \quad (2)$

Let $u = sinx$ in the 2nd integral.

Then (2) $= -\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosec^2x - 1\, dx + \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^2} du$

$= -[-cotx - x]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \left[-\frac{1}{u}\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}}$

$[\frac{d}{dt} cotx = \frac{d}{dt}(tanx)^{-1} = -(tanx)^{-2} sec^2x = -cosec^2x\ ]$

$= \left(\frac{1}{\sqrt{3}} + \frac{\pi}{3}\right) - \left(1 + \frac{\pi}{4}\right) - (\frac{2}{\sqrt{3}} - \sqrt{2})$

$= -\frac{1}{\sqrt{3}} + \frac{\pi}{12} - 1 + \sqrt{2}$

Alternative approach

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{1+secx} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cosx}{cosx+1} dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \frac{1}{cosx+1} dx$

$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cosx-1}{cos^2x-1} dx$

$= \frac{\pi}{12} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cosx-1}{sin^2x} dx \quad (3)$

Let $u = sinx$ in the integral.

Then (3) $= \frac{\pi}{12} + \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^2}\, du - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosec^2x\, dx$

$= \frac{\pi}{12} + \left[-\frac{1}{u}\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} - [-cotx]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \frac{\pi}{12} - \frac{2}{\sqrt{3}} + \sqrt{2} + \frac{1}{\sqrt{3}} - 1$

$= \frac{\pi}{12} - \frac{1}{\sqrt{3}} + \sqrt{2} - 1$

[The official 'Hints' includes a typo:

$$\frac{1-\sec x}{1-\sec^2 x} = \frac{1-\sec x}{\tan^2 x}$$

$(1 - sec^2x = -tan^2x)$]

3rd part

$\int_0^{\frac{\pi}{3}} \frac{1}{(1+sinx)^2}\ dx = \int_0^{\frac{\pi}{3}} \frac{(1-sinx)^2}{(1-sin^2x)^2}\ dx$

$= \int_0^{\frac{\pi}{3}} \frac{1-2sinx+sin^2x}{cos^4x}\ dx$

$= \int_0^{\frac{\pi}{3}} \frac{2-2sinx-cos^2x}{cos^4x}\ dx$

$= 2\int_0^{\frac{\pi}{3}} sec^4x\ dx + 2\int_0^{\frac{\pi}{3}} \frac{-sinx}{cos^4x}\ dx - \int_0^{\frac{\pi}{3}} sec^2x\ dx \quad (4)$

Let $u = cosx$ in the 2nd integral.

Then $(4) = 2\int_0^{\frac{\pi}{3}} sec^2x(1 + tan^2x)dx + 2\int_1^{\frac{1}{2}} \frac{1}{u^4}\ du - [tanx]_0^{\frac{\pi}{3}} \quad (5)$

Let $u = tanx$ in the 1st integral.

Then $(5) = 2\int_0^{\sqrt{3}} (1 + u^2)\ du + 2\left[\frac{u^{-3}}{-3}\right]_1^{\frac{1}{2}} - [tanx]_0^{\frac{\pi}{3}}$

$= 2\left[u + \frac{1}{3}u^3\right]_0^{\sqrt{3}} - \frac{2}{3}(8 - 1) - (\sqrt{3} - 0)$

$= 2(\sqrt{3} + \sqrt{3} - 0) - \frac{14}{3} - \sqrt{3} = 3\sqrt{3} - \frac{14}{3}$