# STEP 2019, P1, Q11 - Solution (2 pages; 3/7/2020)

(i) **1st part**

Prob (they make a decision on the $nth$ round)

$= (pq + qp)^{n-1}(pp + qq) = (q^2 + p^2)(2qp)^{n-1}$ , as required.

**2nd part**

Prob (they make a decision on or before the $nth$ round)

$= \sum_{r=1}^{n}(q^2 + p^2)(2qp)^{r-1} = (q^2 + p^2)\frac{1-(2qp)^n}{1-2qp}$ (1)

And $(p + q)^2 = 1^2$, so that $q^2 + p^2 = (p + q)^2 - 2pq = 1 - 2pq$,

so that (1) $= 1 - (2qp)^n$

Result to prove: $qp \leq \frac{1}{4}$

Proof: $qp \leq \frac{1}{4} \Leftrightarrow 4p(1 - p) \leq 1$

$\Leftrightarrow 4p^2 - 4p + 1 \geq 0$

As LHS $= 4(p - \frac{1}{2})^2$, the result is proved.

Hence $1 - (2qp)^n \geq 1 - \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2^n}$, as required.

(ii) Prob (they make a decision on the $1st$ round) $= p^3 + q^3$

Prob (they make a decision on the $2nd$ round)

$=$ 3Prob(HHT on 1st round)Prob(HH on 2nd round)

+3Prob(TTH on 1st round)Prob(TT on 2nd round)

(The multiple of 3 covers the additional possibilities of HTH or THH on the 1st round, etc.)

$= 3p^2q.p^2 + 3q^2p.q^2$

So Prob (they make a decision on or before the $2nd$ round)

$= p^3 + q^3 + 3p^2q.p^2 + 3q^2p.q^2$

$= (p+q)^3 - 3p^2q - 3pq^2 + 3p^4q + 3q^4p$

$= 1 - 3pq(p + q - p^3 - q^3)$

[to show that the Prob. has a lower bound, we may be able to use the result that $pq \leq \frac{1}{4}$]

$= 1 - 3pq(1 - p^3 - q^3)$   (2)

The critical point will be where $1 - 3\left(\frac{1}{4}\right)(1 - p^3 - q^3) = \frac{7}{16}$;

ie $\frac{9}{16} = \frac{3}{4}(1 - p^3 - q^3)$;

$9 = 12(1 - p^3 - q^3)$;

$12(p^3 + q^3) = 3$;

$p^3 + q^3 = \frac{1}{4}$

Then, for (2) to be at least $\frac{7}{16}$, we want to show that $p^3 + q^3 \geq \frac{1}{4}$

$\Leftrightarrow (p+q)^3 - 3p^2q - 3pq^2 \geq \frac{1}{4}$

$\Leftrightarrow 1 - 3pq(p+q) \geq \frac{1}{4}$

$\Leftrightarrow \frac{3}{4} \geq 3pq$

$\Leftrightarrow pq \leq \frac{1}{4}$ , and this result was established in (i).

Thus we have proved that Prob (they make a decision on or before the $2nd$ round) is at least $\frac{7}{16}$.