STEP 2019, P1, Q10 - Solution (2 pages; 6/7/20)

(i) **1st part**

[The Cartesian eq'n can be quoted (according to the Examiner's Report), but is derived here.]

$x = usin\alpha.t,\ y = ucos\alpha.t - \frac{1}{2}gt^2$

Eliminating $t,\ y = ucos\alpha\left(\frac{x}{usin\alpha}\right) - \frac{1}{2}g\left(\frac{x}{usin\alpha}\right)^2$

Then, as the particle passes through the point $(htan\beta, h)$,

$h = ucos\alpha\left(\frac{htan\beta}{usin\alpha}\right) - \frac{1}{2}g\left(\frac{htan\beta}{usin\alpha}\right)^2$

$\Rightarrow 1 - tan\beta c = -\frac{ghtan^2\beta cosec^2\alpha}{2u^2}$

$= -\frac{tan^2\beta(c^2+1)}{k}$

$\Rightarrow kcot^2\beta - kcot\beta.c = -(c^2 + 1)$

$\Rightarrow c^2 + 1 + kcot^2\beta - kcot\beta.c = 0$

or $c^2 - ckcot\beta + 1 + kcot^2\beta = 0$, as required.

(a) **1st part**

The sum of the roots of the quadratic in c is $-(-kcot\beta)$,

so that $cot\alpha_1 + cot\alpha_2 = kcot\beta$, as required. (1)

2nd part

The product of the roots of the quadratic in c is $1 + kcot^2\beta$,

so that $cot\alpha_1.cot\alpha_2 = 1 + kcot^2\beta$

and $cot\alpha_1.cot\alpha_2 - 1 = kcot^2\beta$ (2)

Then (2) $\div$ (1) $\Rightarrow cot\beta = \frac{cot\alpha_1.cot\alpha_2 - 1}{cot\alpha_1 + cot\alpha_2}$

$\Rightarrow tan\beta = \frac{cot\alpha_1 + cot\alpha_2}{cot\alpha_1 . cot\alpha_2 - 1}$

$= \frac{tan\alpha_2 + tan\alpha_1}{1 - tan\alpha_2 tan\alpha_1} = \tan(\alpha_1 + \alpha_2)$

$\Rightarrow \beta = \alpha_1 + \alpha_2 + 180n$ (for some integer n)

As α_1 & α_2 are positive, and $0 < \beta < 90, n = 0$;

ie $\beta = \alpha_1 + \alpha_2$, as required.

(b) [An inequality in conjunction with a quadratic eq'n suggests the use of the discriminant.]

As there are 2 distinct roots of the quadratic,

$(-kcot\beta)^2 - 4(1 + kcot^2\beta) > 0$

$\Rightarrow cot^2\beta(k^2 - 4k) > 4$

$\Rightarrow k^2 - 4k > 4tan^2\beta$

$\Rightarrow (k-2)^2 - 4 > 4(sec^2\beta - 1)$

$\Rightarrow (k-2)^2 > 4sec^2\beta$

$\Rightarrow k - 2 > 2sec\beta$, provided $k - 2 > 0$ (3)

ie result to prove: $\frac{2u^2}{gh} - 2 > 0$, or $u^2 > gh$

From "$v^2 = u^2 + 2as$", if H is the maximum height reached, then

$0 = (ucos\alpha)^2 - 2gH$, so that $u^2 = \frac{2gH}{cos^2\alpha} > \frac{gh}{1}$, as required (as $\alpha > 0$, so that $cos\alpha < 1$) (4)

Thus, from (3), $k > 2(1 + sec\beta)$, as required.

(ii) From (4), $u^2 = 2gHsec^2\alpha$,

so that $k = \frac{2u^2}{gh} = \frac{4Hsec^2\alpha}{h} \geq 4sec^2\alpha$ (as $H \geq h$), as required.