STEP 2018, P3, Q3 - Solution (2 pages; 24/5/19)

$x^a(x^b(x^cy)')' = x^a(bx^{b-1}(x^cy)' + x^b(x^cy)'')$

$= bx^{a+b-1}[cx^{c-1}y + x^cy'] + x^{a+b}(cx^{c-1}y + x^cy')')$

$= bcx^{a+b+c-2}y + bx^{a+b+c-1}y'$

$+x^{a+b}[c(c-1)x^{c-2}y + cx^{c-1}y' + cx^{c-1}y' + x^cy'']$

$= y''x^{a+b+c} + y'x^{a+b+c-1}[b+2c]$

$+yx^{a+b+c-2}[bc + c(c-1)]$

Comparing with the given DE,

$a+b+c = 2, b+2c = 1-2p, c(b+c-1) = p^2 - q^2$

Let $d = b+c$, so that

$a+d = 2 \ \ (1), c+d = 1-2p \ (2), c(d-1) = p^2-q^2 (3)$

Then, from (2), $d-1 = -c-2p$ and then from (3):

$c(-c-2p) = p^2 - q^2$

or $c^2 + 2pc + p^2 - q^2 = 0$,

so that we can have $c = \frac{-2p+\sqrt{4p^2-4(p^2-q^2)}}{2} = -p+q$

with $b = d-c = (1-c-2p) - c = 1-2(-p+q) - 2p$

$= 1-2q$

and $a = 2-d = 2-(1-c-2p) = 1+(-p+q)+2p$

$= 1+p+q$

(i) The DE can be written as

$x^{1+p+q}(x^{1-2q}(x^{-p+q}y)')' = 0$

$\Rightarrow x^{1-2q}(x^{-p+q}y)' = C$ (in order to hold for all x)

$\Rightarrow x^{-p+q}y = C\int x^{2q-1}dx = \frac{C}{2q}x^{2q} + D$, provided $q \neq 0$

$\Rightarrow y = \frac{C}{2q}x^{p+q} + Dx^{p-q}$ or $C'x^{p+q} + Dx^{p-q}$

If $q = 0$, $y = Cx^p(lnx + D)$

(ii) The DE to solve is $x^{1+p}(x(x^{-p}y)')' = x^n$

$\Rightarrow x(x^{-p}y)' = \int x^{n-p-1}dx$

$\Rightarrow (x^{-p}y)' = \frac{1}{x}(\frac{1}{n-p}x^{n-p} + C)$

unless $p = n$, when $(x^{-p}y)' = \frac{1}{x}(lnx + C)$

For $p \neq n$, $x^{-p}y = \frac{1}{n-p}\int x^{n-p-1} + \frac{C}{x}\ dx$

$\Rightarrow y = \frac{x^p}{n-p}\left(\frac{1}{n-p}x^{n-p} + Clnx + D\right) = \frac{1}{(n-p)^2}x^n + \frac{x^p(Clnx+D)}{n-p}$

For $p = n$, $y = x^p\int\frac{lnx}{x}dx + Cx^p\int\frac{1}{x}dx$

For the 1st integral, let $u = lnx$ [as $\int\frac{1}{x}dx = lnx$],

so that $y = x^p\int udu + Cx^p(lnx + D)$

$= x^p[\frac{1}{2}(lnx)^2 + E] + Cx^p(lnx + D)$

$= x^p[\frac{1}{2}(lnx)^2 + Clnx + C']$

[Only two arbitrary constants are expected for a 2nd order DE.]