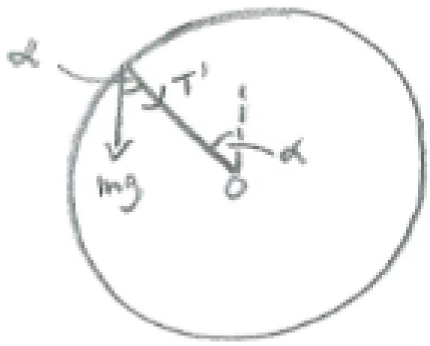
STEP 2018, P3, Q11 - Solution (5 pages; 17/2/21)

1st part



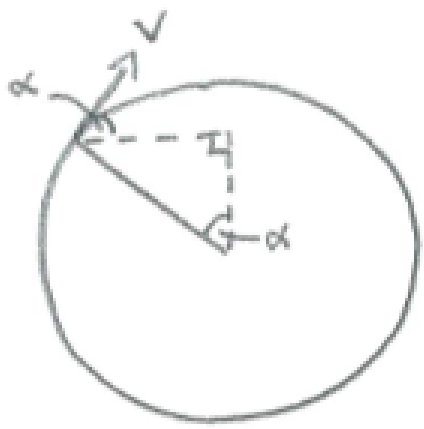
Given that the particle is performing circular motion up to the point when the string is slack, resolving the forces on the particle along the string gives

$T' + mgcos\alpha = m\frac{V^2}{b}$, where m is the mass of the particle

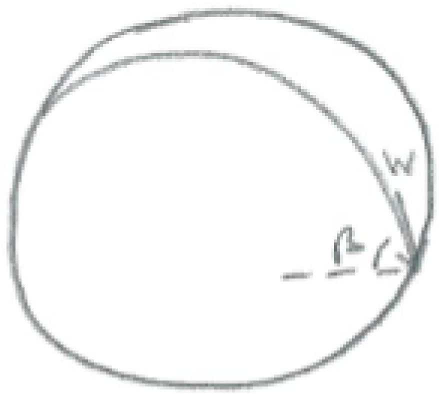
The string becomes slack when the tension, T' in the string becomes zero,

so that $gcos\alpha = \frac{V^2}{b}$, and $V = \sqrt{bgcos\alpha}$ (as $V > 0$)

2nd part



After the string becomes slack, the particle will follow the path of a projectile until it reaches point Q in the diagram. Its initial speed is V, perpendicular to the string, and therefore at an angle α to the horizontal (see the diagram above). Let its speed at Q be W, with its trajectory making an angle β with the horizontal (see the diagram below).



As the horizontal component of the velocity of the particle remains unchanged,

$Vcos\alpha = Wcos\beta$ (1)

Considering the vertical component:

$Vsin\alpha - gT = -Wsin\beta$ (2)

Consider the coordinates of Q, (x, y) relative to O.

The string becomes taut again when $x^2 + y^2 = b^2$ (3)

The coordinates of P (where the string becomes slack) relative to O are $(-bsin\alpha, bcos\alpha)$

And so $bsin\alpha + x = Vcos\alpha.T$ (4)

Also, the (upwards) displacement of the particle in its motion from P to Q (which will be negative) is

(from $'s = \frac{1}{2}(u + v)t'$) $\frac{1}{2}(Vsin\alpha - Wsin\beta)T$,

and so $y = bcos\alpha + \frac{1}{2}(Vsin\alpha - Wsin\beta)T$, which from (2) becomes

$y = bcos\alpha + \frac{1}{2}(2Vsin\alpha - gT)T$ (5)

Substituting for x & y from (4) & (5) into (3) gives

$(Vcos\alpha.T - bsin\alpha)^2 + \left(bcos\alpha + VTsin\alpha - \frac{1}{2}gT^2\right)^2 = b^2$

[This doesn't look very promising, although we can see that $b^2sin^2\alpha + b^2cos^2\alpha = b^2$, and $V^2 = bgcos\alpha$ may help.]

$\Rightarrow V^2cos^2\alpha.T^2 + b^2sin^2\alpha - 2VTbcos\alpha sin\alpha + b^2cos^2\alpha$

$+V^2T^2sin^2\alpha + \frac{1}{4}g^2T^4 + 2bVTcos\alpha sin\alpha - bgT^2cos\alpha$

$-VgT^3sin\alpha = b^2$

$\Rightarrow V^2cos^2\alpha.T^2 + V^2T^2sin^2\alpha + \frac{1}{4}g^2T^4 - bgT^2cos\alpha - VgT^3sin\alpha = 0$

$\Rightarrow V^2T^2 + \frac{1}{4}g^2T^4 - bgT^2cos\alpha - VgT^3sin\alpha = 0$

$\Rightarrow V^2 + \frac{1}{4}g^2T^2 - bgcos\alpha - VgTsin\alpha = 0$ (as $T \neq 0$)

$\Rightarrow \frac{1}{4}g^2T^2 - VgTsin\alpha = 0$

$\Rightarrow gT = 4Vsin\alpha$, as required

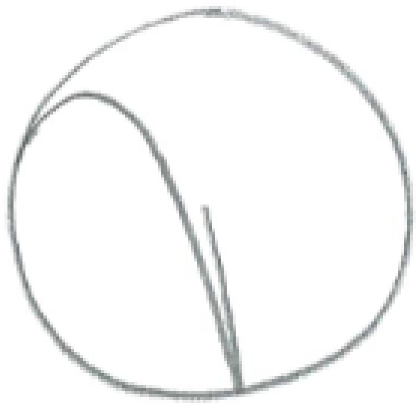
3rd part

Dividing (2) by (1) gives

$\frac{Vsin\alpha - 4Vsin\alpha}{Vcos\alpha} = -tan\beta$,

so that $tan\beta = 3tan\alpha$, as required

4th part



[Obviously calculators can't be used in the STEP exam, but the necessary α and β implied by $sin^2\alpha = \frac{1+\sqrt{3}}{4}$ and $tan\beta = 3tan\alpha$ are $\alpha = 56°$ and $\beta = 77°$ (to the nearest degree).]

The particle will come to rest if it is travelling in the direction of the string (away from O) just before the string becomes taut again. Referring to the diagram, this will occur if

$\frac{-y}{x} = tan\beta = 3tan\alpha$

From (4) & (5), this means that $\frac{-\{bcos\alpha+\frac{1}{2}(2Vsin\alpha-gT)T\}}{Vcos\alpha.T-bsin\alpha} = 3tan\alpha$,

when $gT = 4Vsin\alpha$ (and $V^2 = bgcos\alpha$)

$\Rightarrow \frac{bcos\alpha-Vsin\alpha.T}{bsin\alpha-Vcos\alpha.T} = 3tan\alpha$

$\Rightarrow \frac{bgcos\alpha-Vsin\alpha.gT}{bsin\alpha-Vcos\alpha.T} = 3gtan\alpha$

$\Rightarrow \frac{V^2-Vsin\alpha.4Vsin\alpha}{bsin\alpha-Vcos\alpha.T} = 3gtan\alpha$

$\Rightarrow V^2(1-4sin^2\alpha) = \frac{3gbsin^2\alpha}{cos\alpha} - 3gTVsin\alpha$

$= \frac{3bgcos\alpha sin^2\alpha}{cos^2\alpha} - 3Vsin\alpha(4Vsin\alpha)$

Then, writing $A = sin^2\alpha$,

$V^2(1-4A) = \frac{3\,V^2A}{1-A} - 12V^2A$,

so that (assuming $V \neq 0$) $1 - 4A = \frac{3A}{1-A} - 12A$

$\Rightarrow (1+8A)(1-A) = 3A$

$\Rightarrow 1 - 8A^2 + 7A = 3A$

$\Rightarrow 8A^2 - 4A - 1 = 0$

$\Rightarrow sin^2\alpha = \frac{4 \pm \sqrt{16+32}}{16}$

$= \frac{1+\sqrt{3}}{4}$ (rejecting the negative root), as required