STEP 2018, P2, Q13 - Solution (2 pages; 11/10/18)

(i) $A_1 = \frac{1}{2}$, $B_1 = \frac{1}{4}$, $C_1 = 0$, $D_1 = \frac{1}{4}$

$A_2 = P(LR\ or\ NN\ or\ RL) = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{8}$

[L = left; R = right; N = no movement]

$B_2 = P(NR\ or\ RN) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (\frac{1}{4})(\frac{1}{2}) = \frac{1}{4}$

$C_2 = P(RR\ or\ LL) = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{1}{8}$

$D_2 = B_2$ (by symmetry) $= \frac{1}{4}$

(ii) Observations: $D_n = B_n$, by symmetry

and $A_n + B_n + C_n + D_n = 1$

Now, $B_{n+1} + D_{n+1} = (A_n\left(\frac{1}{4}\right) + B_n\left(\frac{1}{2}\right) + C_n\left(\frac{1}{4}\right))$

$+(A_n\left(\frac{1}{4}\right) + D_n\left(\frac{1}{2}\right) + C_n\left(\frac{1}{4}\right))$

$= \frac{1}{2}(A_n + B_n + C_n + D_n) = \frac{1}{2}$,

so that $B_n = D_n = \frac{1}{4}$

Then $A_n + C_n = 1 - (B_n + D_n) = \frac{1}{2}$ (1)

and $A_{n+1} = D_n\left(\frac{1}{4}\right) + A_n\left(\frac{1}{2}\right) + B_n\left(\frac{1}{4}\right)$

$= \frac{1}{2}A_n + \frac{1}{8}$

Thus $A_2 = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{8}$, $A_3 = \frac{1}{2}[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{8}] + \frac{1}{8}$

$A_4 = \frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{8}\right] + \frac{1}{8}\right\} + \frac{1}{8}$

$= \frac{1}{8}\left\{1 + \frac{1}{2} + \frac{1}{2^2}\right\} + \frac{1}{2^4}$

and $A_n = \frac{1}{8}\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}}\right\} + \frac{1}{2^n}$

$= \frac{1}{8}\frac{(1-\left(\frac{1}{2}\right)^{n-1})}{1-\frac{1}{2}} + \frac{1}{2^n}$

$= \frac{1}{4}\{1 - \left(\frac{1}{2}\right)^{n-1}\} + \frac{1}{2^n}$

$= \frac{1}{4}\{1 - \left(\frac{1}{2}\right)^{n-1} + 2\left(\frac{1}{2}\right)^{n-1}\}$

$= \frac{1}{4}\{1 + \left(\frac{1}{2}\right)^{n-1}\}$

and hence, from (1), $C_n = \frac{1}{2} - \frac{1}{4}\{1 + \left(\frac{1}{2}\right)^{n-1}\}$

$= \frac{1}{4}\{1 - \left(\frac{1}{2}\right)^{n-1}\}$