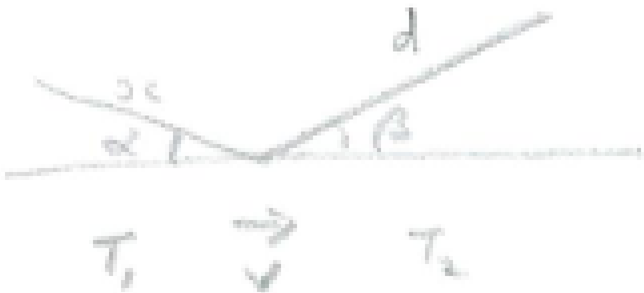
STEP 2018, Paper 1, Q9 – Solution (2 pages; 29/1/21)

(i) **1st part**



If the go-kart reaches the house, its final potential energy will be no greater than its initial potential energy. Hence, referring to the diagram,

$mgdsin\beta \leq mgxsin\alpha$, where m is the mass of the go-kart;

so that $xsin\alpha \geq dsin\beta$, as required

2nd part

Let T_1 & T_2 be the times taken for the 2 parts of the journey, and let v be the speed at the traffic lights.

Then, applying suvat eq'ns:

$x = \frac{1}{2}(gsin\alpha)T_1^{\ 2}$ (1)

$d = vT_2 - \frac{1}{2}(gsin\beta)T_2^{\ 2}$ (2)

$v = gsin\alpha T_1$ (3)

Then (2) & (3) $\Rightarrow \frac{1}{2}(gsin\beta)T_2^{\ 2} - gsin\alpha T_1 T_2 + d = 0$ (4)

And (1) & (4) $\Rightarrow \frac{1}{2}(gsin\beta)T_2^{\ 2} - gsin\alpha\sqrt{\frac{2x}{gsin\alpha}}T_2 + d = 0$

$$\Rightarrow T_2 = \frac{\sqrt{2xgsin\alpha} \pm \sqrt{2xgsin\alpha - 2gsin\beta d}}{gsin\beta} \quad (5)$$

However, the larger value can be excluded, as it relates to when the go-kart has continued beyond the house and then returned back to it (having come to a halt on the slope).

Also, from (1), $T_1 = \sqrt{\frac{2x}{gsin\alpha}}$

The total time $T = T_1 + T_2$

$$= \sqrt{\frac{2x}{gsin\alpha}} + \frac{\sqrt{2xgsin\alpha} - \sqrt{2xgsin\alpha - 2gsin\beta d}}{gsin\beta}$$

$$\Rightarrow \left(\frac{gsin\alpha}{2}\right)^{\frac{1}{2}} T = \sqrt{x} + \sqrt{\frac{gsin\alpha}{2} \cdot \frac{2xgsin\alpha}{g^2 sin^2\beta}}$$

$$- \sqrt{\frac{gsin\alpha}{2} \cdot \frac{(2xgsin\alpha - 2gsin\beta d)}{g^2 sin^2\beta}}$$

$= \sqrt{x} + k\sqrt{x} - \sqrt{k^2 x - kd}$, where $k = \frac{sin\alpha}{sin\beta}$

$= (1 + k)\sqrt{x} - \sqrt{k^2 x - kd}$, as required.

3rd part

$$\frac{dT}{dx} = 0 \Rightarrow \frac{1}{2}(1 + k)x^{-\frac{1}{2}} - \frac{1}{2}(k^2 x - kd)^{-\frac{1}{2}} k^2 = 0$$

$$\Rightarrow (1 + k)^2 x^{-1} = (k^2 x - kd)^{-1} k^4$$

$$\Rightarrow k^4 x = (1 + k)^2 (k^2 x - kd)$$

$$\Rightarrow k^3 x = (1 + k)^2 (kx - d)$$

$$\Rightarrow x\left(k^3 - (k + 2k^2 + k^3)\right) = -(1 + k)^2 d$$

$$\Rightarrow x = \frac{(1+k)^2 d}{k + 2k^2} \text{ or } \frac{(1+k)^2 d}{k(1+2k)}$$