STEP 2018, P1, Q7 - Solution (3 pages; 9/5/20)

(i) With $x = \frac{pz+q}{z+1}$, $x^3 - 3pqx + pq(p+q) = 0$ becomes

$(\frac{pz+q}{z+1})^3 - 3pq(\frac{pz+q}{z+1}) + pq(p+q) = 0$

$\Rightarrow (pz+q)^3 - 3pq(pz+q)(z+1)^2 + pq(p+q)(z+1)^3 = 0$ (A)

Coeff. of z^3 on LHS of (A) is

$p^3 - 3pq.p + pq(p+q) = p(p^2 - 3pq + pq + q^2)$

$= p(p-q)^2$

Coeff. of z^2 on LHS of (A) is

$3p^2q - 3pq(2p+q) + pq(p+q)(3)$

$= pq(3p - 6p - 3q + 3p + 3q) = 0$

Coeff. of z on LHS of (A) is $3pq^2 - 3pq(p+2q) + pq(p+q)(3)$

$= pq(3q - 3p - 6q + 3p + 3q) = 0$

Constant term on LHS of (A) is $q^3 - 3pq.q + pq(p+q)$

$= q(q^2 - 2pq + p^2) = q(q-p)^2$

Thus the equation (A) reduces to $p(p-q)^2z^3 + q(p-q)^2 = 0$

and, as $p \neq q$, $pz^3 + q = 0$

(ii) Suppose that $c = pq$ and $d = pq(p+q)$ where $p \neq q$

Then $d = c(p+q)$ and so $d = c(p + \frac{c}{p})$

and $dp = cp^2 + c^2$, so that $cp^2 - dp + c^2 = 0$ (B)

This has distinct real sol'ns when $d^2 - 4c^3 > 0$

and, by symmetry, the sol'ns of (B) will be p & q.

Thus, provided that $d^2 > 4c^3$, there will be distinct solutions p & q of $cx^2 - dx + c^2 = 0$, with $pq = \frac{c^2}{c} = c$

and $p + q = \frac{-(-d)}{c}$, so that $d = pq(p+q)$;

ie c & d can be expressed in terms of p & q, as required.

[The Examiner's Report indicates that candidates need to be careful to show that $c = pq$ and $d = pq(p+q)$ is possible, provided that $d^2 > 4c^3$; rather than showing that $d^2 > 4c^3$ when $c = pq$ and $d = pq(p+q)$.]

(iii) $x^3 + 6x - 2 = 0$ can be written in the form

$x^3 - 3pqx + pq(p+q) = 0$,

where p & q are the roots of $cx^2 - dx + c^2 = 0$, from (B) in (ii),

where $c = -2$ & $d = -2$

So $-2x^2 + 2x + 4 = 0$, or $x^2 - x - 2 = 0$,

so that $(x-2)(x+1) = 0$ and hence $p = 2, q = -1$ (say).

Thus, from (i), if $x = \frac{pz+q}{z+1} = \frac{2z-1}{z+1}$,

then , $pz^3 + q = 0$; ie $2z^3 - 1 = 0$,

with real root $z = 2^{-\frac{1}{3}}$,

giving $x = \frac{2z-1}{z+1} = \frac{2(2^{-\frac{1}{3}})-1}{(2^{-\frac{1}{3}})+1} = \frac{2-2^{\frac{1}{3}}}{1+2^{\frac{1}{3}}}$

(iv) $x^3 - 3p^2x + 2p^3 = 0$

$\Rightarrow (x - p)(x^2 + px - 2p^2) = 0$

$\Rightarrow (x - p)(x - p)(x + 2p) = 0$

So roots are $p, p \,\&\, -2p$. (C)

Consider $x^3 - 3cx + d = 0$, where $d^2 = 4c^3$

From the working to (ii), we can write $c = p^2 \,\&\, d = 2p^3$,

to give $x^3 - 3p^2x + 2p^3 = 0$

with roots $p, p \,\&\, -2p$, from (C).

Thus the roots of $x^3 - 3cx + d = 0$ (with $d^2 = 4c^3$) are:

$\sqrt{c}, \sqrt{c} \,\&\, -2\sqrt{c}$ or $-\sqrt{c}, -\sqrt{c} \,\&\, 2\sqrt{c}$

If $\sqrt{c}$ is a root, then $c\sqrt{c} - 3c(\sqrt{c}) + d = 0$, so that $d = 2c\sqrt{c}$,

whilst if $-\sqrt{c}$ is a root, then $-c\sqrt{c} - 3c(-\sqrt{c}) + d = 0$,

so that $d = -2c\sqrt{c}$