STEP 2018, P1, Q13 - Solution (2 pages; 16/5/20)

(i) If $k < 3$, then A cannot pass.

Let X be the number of correct answers. Then $X \sim B(k, \frac{1}{n})$

If $k = 3$, then A will pass only if $X = 3$.

$P(X = 3) = (\frac{1}{n})^3$ (A)

If $k = 4$, then A will pass only if $X = 3$ or 4

$P(X = 3 \; or \; 4) = 4\left(\frac{1}{n}\right)^3\left(1 - \frac{1}{n}\right) + (\frac{1}{n})^4$

$= \left(\frac{1}{n}\right)^3\left\{4 - \frac{4}{n} + \frac{1}{n}\right\} = \left(\frac{1}{n}\right)^3\frac{(4n-3)}{n} = \frac{4n-3}{n^4}$ (B)

To compare with (A):

$\frac{(4n-3)}{n^4} - \frac{1}{n^3} = \frac{3n-3}{n^4} = \frac{3(n-1)}{n^4} > 0$, as $n \geq 2$, so (B) $>$ (A)

If $k = 5$, then A will pass only if $X = 4$ or 5

$P(X = 4 \; or \; 5) = 5\left(\frac{1}{n}\right)^4\left(1 - \frac{1}{n}\right) + (\frac{1}{n})^5$

$= \left(\frac{1}{n}\right)^3\left\{\frac{5}{n} - \frac{5}{n^2} + \frac{1}{n^2}\right\} = \left(\frac{1}{n}\right)^3\frac{(5n-4)}{n^2} = \frac{5n-4}{n^5}$ (C)

To compare with (B):

$\frac{(4n-3)}{n^4} - \frac{(5n-4)}{n^5} = \frac{4n^2-3n-5n+4}{n^5} = \frac{4\left(n^2-2n+1\right)}{n^2} = \frac{4(n-1)^2}{n^2} > 0$

when $n \geq 2$

So (B) $>$ (C), and the probability of passing is maximised when $k = 4$.

(ii) $P(k=4 \mid B\ passes) = \frac{P(k=4\ \&\ B\ passes)}{P(B\ passes)}$

$P(k=4\ \&\ B\ passes) = P(B\ passes\ |k=4)P(k=4)$

$= \frac{4n-3}{n^4}\ .\frac{1}{6} = \frac{4n-3}{6n^4}$

and $P(B\ passes) = \frac{1}{6}P(passes\ |k=3) + \frac{1}{6}P(passes\ |k=4)$

$+\frac{1}{6}P(passes\ |k=5)$

$= \frac{1}{6}(\frac{1}{n^3} + \frac{4n-3}{n^4} + \frac{5n-4}{n^5}\}$

$= \frac{n^2+n(4n-3)+(5n-4)}{6n^5} = \frac{5n^2+2n-4}{6n^5}$

So $P(k=4 \mid B\ passes) = \frac{\left(\frac{4n-3}{6n^4}\right)}{(\frac{5n^2+2n-4}{6n^5})} = \frac{n(4n-3)}{5n^2+2n-4}$

(iii) $P(C\ passes) = P(C\ answers\ 3\ q'ns)\frac{1}{n^3}$

$+P(C\ answers\ 4\ q'ns)\frac{4n-3}{n^4}$

$+P(C\ answers\ 5\ q'ns)\frac{5n-4}{n^5}$

$= \binom{5}{3}(\frac{n}{n+1})^3(1-\frac{n}{n+1})^2\frac{1}{n^3}$

$+5\left(\frac{n}{n+1}\right)^4(1-\frac{n}{n+1})\frac{4n-3}{n^4}$

$+\left(\frac{n}{n+1}\right)^5\frac{5n-4}{n^5}$

$= \frac{1}{(n+1)^5}\{10+5(4n-3)+(5n-4)\} = \frac{25n-9}{(n+1)^5}$