STEP 2018, P1, Q12 - Solution (3 pages; 14/5/20)

(i) $P(Head) = \sum_{i=1}^{3} P(coin\ i\ is\ drawn) \times P(coin\ i\ shows\ Head)$

$= \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 = \frac{1}{3}(p_1 + p_2 + p_3)$

(ii) $[N_1 \sim B(2, p) \Rightarrow E(N_1) = 2p$ & $Var(N_1) = 2p(1-p)$ and, according to the Examiner's Report, it was acceptable to just quote theseust be quoted.]

$E(N_1) = 1. P(1\ Head\ from\ 2\ coins)$

$+2. (2\ Heads\ from\ 2\ coins)$

$= 2p(1-p) + 2p^2 = 2p$

$E({N_1}^2) = 1^2. P(1\ Head\ from\ 2\ coins)$

$+2^2. P(2\ Heads\ from\ 2\ coins)$

$= 2p(1-p) + 4p^2 = 2p + 2p^2$

$Var(N_1) = E({N_1}^2) - [E(N_1)]^2$

$= 2p + 2p^2 - 4p^2$

$= 2p(1-p)$

(iii) $E(N_2) = 1. P(1\ Head\ from\ the\ 2\ coins)$

$+2. (2\ Heads\ from\ the\ 2\ coins)$

$= \{P(Coins\ 1\ \&\ 2\ are\ chosen). (p_1(1-p_2) + (1-p_1)p_2)$

$+P(Coins\ 1\ \&\ 3\ are\ chosen). (p_1(1-p_3) + (1-p_1)p_3)$

$+P(Coins\ 2\ \&\ 3\ are\ chosen). (p_2(1-p_3) + (1-p_2)p_3)\}$

$+2\{P(Coins\ 1\ \&\ 2\ are\ chosen).\, p_1 p_2$

$+P(Coins\ 1\ \&\ 3\ are\ chosen).\, p_1 p_3$

$+P(Coins\ 2\ \&\ 3\ are\ chosen).\, p_2 p_3\}$

$= \{\frac{1}{3}(p_1(1-p_2) + (1-p_1)p_2)$

$+\frac{1}{3}(p_1(1-p_3) + (1-p_1)p_3)$

$+\frac{1}{3}(p_2(1-p_3) + (1-p_2)p_3)\}$

$+2\{\frac{1}{3}p_1p_2 + \frac{1}{3}p_1p_3 + \frac{1}{3}p_2p_3\}$

$= \frac{1}{3}(2p_1 + 2p_2 + 2p_3) = 2p$

$E\left({N_2}^2\right) = \{\frac{1}{3}(p_1(1-p_2) + (1-p_1)p_2)$

$+\frac{1}{3}(p_1(1-p_3) + (1-p_1)p_3)$

$+\frac{1}{3}(p_2(1-p_3) + (1-p_2)p_3)\}$

$+4\{\frac{1}{3}p_1p_2 + \frac{1}{3}p_1p_3 + \frac{1}{3}p_2p_3\}$

$= \frac{1}{3}(2p_1 + 2p_2 + 2p_3 + 2p_1p_2 + 2p_1p_3 + 2p_2p_3)$

$= 2p + \frac{2}{3}(p_1p_2 + p_1p_3 + p_2p_3)$

$Var(N_2) = E\left({N_2}^2\right) - [E(N_2)]^2$

$= 2p + \frac{2}{3}(p_1p_2 + p_1p_3 + p_2p_3) - 4p^2$

(iv) $Var(N_1) - Var(N_2) =$

$= 2p(1-p) - \{2p + \frac{2}{3}(p_1p_2 + p_1p_3 + p_2p_3) - 4p^2\}$

$= 2p^2 - \frac{2}{3}(p_1p_2 + p_1p_3 + p_2p_3)$

$= \frac{2}{9}(p_1 + p_2 + p_3)^2 - \frac{2}{3}(p_1p_2 + p_1p_3 + p_2p_3)$

$= \frac{2}{9}({p_1}^2 + {p_2}^2 + {p_3}^2) - \frac{2}{9}(p_1p_2 + p_1p_3 + p_2p_3)$

$= \frac{1}{9}((p_1 - p_2)^2 + (p_1 - p_3)^2 + (p_2 - p_3)^2)$,

which is positive, unless $p_1 = p_2 = p_3$,

giving $Var(N_2) \leq Var(N_2)$, as required.