STEP 2017, P3, Q9 - Solution (2 pages; 14/7/20)

1st part

[Note that, because the partices are connected by a spring, rather than an inextensible string, the accelerations of A and B will not be constant, and so suvat cannot be used.]

Applying N2L to A: $mg - T = m\ddot{y}$,

where T is the tension in the spring.

Applying N2L to B: $T = 2m\ddot{x}$

Eliminating T, $mg - 2m\ddot{x} = m\ddot{y}$,

so that $g - 2\ddot{x} = \ddot{y}$

Then, integrating wrt t:

$gt - 2\dot{x} = \dot{y} + C$; and when $t = 0, \dot{x} = \dot{y} = 0$, so that $C = 0$

And integrating again:

$\frac{1}{2}gt^2 - 2x = y + D$; and when $t = 0, x = y = 0$, so that $D = 0$

So $y + 2x = \frac{1}{2}gt^2$, as required.

2nd part

Taking the zero of GPE as being at the top of the table,

the initial total energy is 0 (as the spring is at its natural length).

At time T:

GPE of B is 0

GPE of A is $-mgy(T) = -mg\left(\frac{1}{2}gT^2 - 2x(T)\right)$

$= -mg(\frac{1}{2}g\left(\frac{6a}{g}\right) - 2a)$

$= -mga$

KE of B is $\frac{1}{2}(2m)v^2 = mv^2$, where v is the speed to be found

KE of A is $\frac{1}{2}m(\dot{y}(T))^2$,

and $y + 2x = \frac{1}{2}gt^2 \Rightarrow \dot{y} + 2\dot{x} = gt + E$;

and when $t = 0, \dot{x} = \dot{y} = 0$, so that $E = 0$

Hence $\dot{y}(T) + 2v = gT$,

and KE of A is $\frac{1}{2}m(gT - 2v)^2$

Also, Elastic PE (at time T) is: $\frac{1}{2}\left(\frac{\lambda}{a}\right)(y(T) - a)^2$

and $y + 2x = \frac{1}{2}gt^2 \Rightarrow y(T) + 2a = \frac{1}{2}g\left(\frac{6a}{g}\right)$,

so that $y(T) = a$, and EPE at time T is 0

Then (GPE of B)+(GPE of A)+(KE of B)+(KE of A)+EPE $= 0$,

so that $-mga + mv^2 + \frac{1}{2}m(gT - 2v)^2 = 0$,

and $-2ag + 2v^2 + (g^2T^2 - 4gTv + 4v^2) = 0$

$\Rightarrow 6v^2 - 4gTv + g^2\left(\frac{6a}{g}\right) - 2ag = 0$

$\Rightarrow 3v^2 - 2gTv + 2ag = 0$

$\Rightarrow v = \frac{2gT \pm \sqrt{4g^2T^2 - 24ag}}{6}$

Then $4g^2T^2 - 24ag = 4g^2\left(\frac{6a}{g}\right) - 24ag = 0$,

so that $v = \frac{2g\sqrt{6a/g}}{6} = \sqrt{2ag/3}$, as required.