STEP 2017, P3, Q4 - Solution (2 pages; 15/7/20)

(i) Let $a = e^b$ $(a \neq 1 \Rightarrow b \neq 0)$

Then $log_a f(x) = log_a e . ln f(x) = \frac{1}{lna} . lnf(x) = \frac{1}{b} lnf(x)$

(as $b \neq 0, \frac{1}{b}$ is defined)

Then $a^{\frac{1}{y}\int_0^y log_a f(x)dx} = e^{\frac{b}{y}\int_0^y \frac{1}{b} lnf(x)dx}$

$= e^{\frac{1}{y}\int_0^y lnf(x)dx} = F(y)$, as required.

(ii) $F(y)G(y) = e^{\frac{1}{y}\int_0^y lnf(x)dx} . e^{\frac{1}{y}\int_0^y lng(x)dx}$

$= e^{\frac{1}{y}\int_0^y lnf(x)+lng(x)dx}$

$= e^{\frac{1}{y}\int_0^y ln(f(x)g(x))dx}$

$= e^{\frac{1}{y}\int_0^y ln\, h(x)dx}$

$= H(y)$, as required.

(iii) Let $f(x) = b^x$ (so that $f(x) > 0$, as $b > 0$)

Then $F(y) = e^{\frac{1}{y}\int_0^y ln\, b^x dx}$

$= e^{\frac{1}{y}\int_0^y xlnb\, dx}$

$= e^{\frac{lnb}{y}[\frac{1}{2}x^2]_0^y}$

$= b^{\frac{1}{y}(\frac{1}{2}y^2)}$

$= b^{\frac{y}{2}}$ or $\sqrt{b^y}$

(iv) $e^{\frac{1}{y}\int_0^y ln\, f(x)dx} = \sqrt{f(y)}$

$\Rightarrow \frac{1}{y}\int_0^y ln\, f(x)dx = ln\sqrt{f(y)} = \frac{1}{2}lnf(y)$

$\Rightarrow \int_0^y ln\, f(x)dx = \frac{y}{2}lnf(y)$

Differentiating wrt y,

$lnf(y) = \frac{1}{2}lnf(y) + \frac{y}{2} \cdot \frac{1}{f(y)}\, f'(y)$

$\Rightarrow lnf(y) = \frac{y}{f(y)}\, f'(y) \quad (1)$

Let $f(x) = e^{g(x)}$ (as $f(x) > 0$)

Result to prove: $g(x) = cx$, so that $f(x) = e^{cx} = b^x$,

where $b = e^c (> 0)$

Then (1) $\Rightarrow g(y) = \frac{y}{e^{g(y)}} \cdot e^{g(y)} g'(y)$

$\Rightarrow g(y) = yg'(y)$

$\Rightarrow \frac{1}{y} = \frac{g'(y)}{g(y)}$

$\Rightarrow \int \frac{1}{y}\, dy = \int \frac{g'(y)}{g(y)}\, dy$

$\Rightarrow \ln(cy) = \ln(g(y))$

$\Rightarrow cy = g(y)$, or $g(x) = cx$, as required.