STEP 2017, P3, Q13 - Solution (2 pages; 17/2/21)

1st part

$V(x) = E((X-x)^2) = E(X^2 + x^2 - 2Xx)$

$= E(X^2) + x^2 - 2xE(X)$

Now $\sigma^2 = E(X^2) - \mu^2$,

so that $V(x) = (\sigma^2 + \mu^2) + x^2 - 2x\mu$

$= \sigma^2 + (x-\mu)^2$

2nd part

$E(Y) = E(\sigma^2 + (X-\mu)^2) = \sigma^2 + \sigma^2 = 2\sigma^2$, as required

3rd part

$\mu = \frac{1}{2}(0+1) = \frac{1}{2}$ and [from the Formulae booklet - no longer provided (@ Feb. 2021)] $\sigma^2 = \frac{1}{12}(1-0)^2 = \frac{1}{12}$,

so that $V(x) = \sigma^2 + (x-\mu)^2 = \frac{1}{12} + (x-\frac{1}{2})^2$ or $x^2 - x + \frac{1}{3}$

4th part

Consider $P(Y \leq y) = P(\frac{1}{12} + (X - \frac{1}{2})^2 \leq y)$

(Note that $\frac{1}{12} \leq \frac{1}{12} + (X-\frac{1}{2})^2 \leq \frac{1}{12} + \left(1-\frac{1}{2}\right)^2 = \frac{1}{3}$)

$= P(-\sqrt{y - \frac{1}{12}} \leq x - \frac{1}{2} \leq \sqrt{y - \frac{1}{12}})$

$= P(\frac{1}{2} - \sqrt{y - \frac{1}{12}} \leq x \leq \frac{1}{2} + \sqrt{y - \frac{1}{12}})$

[As a check, $\frac{1}{2} - \sqrt{y - \frac{1}{12}} \geq 0$ and $\frac{1}{2} + \sqrt{y - \frac{1}{12}} \leq 1$ means that, in both cases, $0 \leq y - \frac{1}{12} \leq \frac{1}{4}$; ie $\frac{1}{12} \leq y \leq \frac{1}{3}$]

$$= \left(\frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) - \left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) = 2\sqrt{y - \frac{1}{12}} \quad \left(\text{for } \frac{1}{12} \leq y \leq \frac{1}{3}\right)$$

[As a further check, $2\sqrt{\frac{1}{3} - \frac{1}{12}} = 1$]

Then the pdf of Y is $\frac{d}{dy}\left(2\sqrt{y - \frac{1}{12}}\right) = \frac{1}{\sqrt{y - \frac{1}{12}}}$ for $\frac{1}{12} \leq y \leq \frac{1}{3}$

(and zero elsewhere)

5th part

To verify that $E(Y) = 2\sigma^2 = 2\left(\frac{1}{12}\right) = \frac{1}{6}$ in this case:

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} \frac{y}{\sqrt{y - \frac{1}{12}}} dy$$

Let $u = y - \frac{1}{12}$, so that $E(Y) = \int_0^{\frac{1}{4}} \frac{u + \frac{1}{12}}{\sqrt{u}} \, du$

$$= \left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + \frac{u^{\frac{1}{2}}}{12\left(\frac{1}{2}\right)}\right]_0^{\frac{1}{4}} = \frac{2}{3}\left(\frac{1}{8}\right) + \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{6}, \text{ as required}$$