STEP 2017, P2, Q1 - Solution (2 pages; 14/10/19)

(i) By Parts, $I_n = \int_0^1 x^n arctanx\, dx$

$= \left[\frac{x^{n+1}}{n+1} arctanx\right]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\left(\frac{1}{1+x^2}\right) dx$ (for $n \geq 0$)

$= \frac{(\frac{\pi}{4})}{n+1} - 0 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2} dx$

so that $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} dx$, as required

And $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2}[\ln(1+x^2)]_0^1$

$= \frac{\pi}{4} - \frac{1}{2} ln2$

(ii) From (i), $(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+3}+x^{n+1}}{1+x^2} dx$

$= \frac{\pi}{2} - \int_0^1 x^{n+1} dx = \frac{\pi}{2} - \left[\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\pi}{2} - \frac{1}{n+2}$

So $3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$ and $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$

Then $5I_4 + \left(\frac{\pi}{2} - \frac{1}{2} - I_0\right) = \frac{\pi}{2} - \frac{1}{4}$

$\Rightarrow 5I_4 = \frac{1}{4} + I_0 = \frac{1}{4} + (\frac{\pi}{4} - \frac{1}{2} ln2)$, from (i)

$\Rightarrow I_4 = \frac{1}{20}(1 + \pi - 2ln2)$

(iii) When $n = 1$, result to prove: $5I_4 = A - \frac{1}{2}\sum_{r=1}^{2}(-1)^r.\frac{1}{r}$

$= A - \frac{1}{2}\left(-1 + \frac{1}{2}\right) = A + \frac{1}{4}$; ie $I_4 = \frac{1}{20}(1 + 4A)$

From (ii), the result is true when $\pi - 2ln2 = 4A$;

ie $A = \frac{1}{4}(\pi - 2ln2)$

Suppose that the result is true for $n = k$, so that

$(4k+1)I_{4k} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r.\frac{1}{r}$ (1)

Target result for $n = k+1$: $(4k+5)I_{4k+4} = A - \frac{1}{2}\sum_{r=1}^{2k+2}(-1)^r.\frac{1}{r}$

From (ii), $(4k+5)I_{4k+4} + (4k+3)\ I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4}$ (2)

and also $(4k+3)I_{4k+2} + (4k+1)\ I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2}$ (3)

Then, $(2) - (3)$ gives

$(4k+5)I_{4k+4} - (4k+1)\ I_{4k} = \frac{1}{4k+2} - \frac{1}{4k+4}$

Then, from (1), $(4k+5)I_{4k+4} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r.\frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$

$= A - \frac{1}{2}\sum_{r=1}^{2k+2}(-1)^r.\frac{1}{r} + \frac{1}{2}\left\{(-1)^{2k+1}.\frac{1}{2k+1} + (-1)^{2k+2}.\frac{1}{2k+2}\right\} + \frac{1}{4k+2} - \frac{1}{4k+4}$

$$= A - \frac{1}{2}\sum_{r=1}^{2k+2}(-1)^r.\frac{1}{r} + \frac{1}{2}\{-\frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+1} - \frac{1}{2k+2}\}$$

$= A - \frac{1}{2}\sum_{r=1}^{2k+2}(-1)^r.\frac{1}{r}$, as required

So, if the result is true for $n = k$, then it is true for $n = k+1$

As it is true for $n = 1$, it will therefore be true for $n = 2,3, \ldots$ and hence all integer $n \geq 1$, by the principle of induction.