STEP 2017, P2, Q13 - Sol'n (2 pages; 5/6/20)

(i) $E(X) = 1\left(\frac{1}{n}\right) + 2\left(\frac{n-1}{n}\right)\left(\frac{1}{n}\right) + \cdots + k\left(\frac{n-1}{n}\right)^{k-1}\left(\frac{1}{n}\right) + \cdots$

$= \frac{1}{n}\sum_{k=1}^{\infty} k\left(\frac{n-1}{n}\right)^{k-1}$ (1)

Consider $(1-q)^{-2} = 1 + (-2)(-q) + \frac{(-2)(-3)}{2!}(-q)^2$

$+ \frac{(-2)(-3)(-4)}{3!}(-q)^3$

$= 1 + 2q + 3q^2 + 4q^3 + \cdots$

Let $q = \frac{n-1}{n}$, so that $E(X) = \frac{1}{n}\left(1 - \frac{n-1}{n}\right)^{-2}$

$= \frac{1}{n}\left(\frac{1}{n}\right)^{-2} = n$

[Alternatively, $(1) = \frac{1}{n}\sum_{k=1}^{\infty} k\lambda^{k-1}$, where $\lambda = \frac{n-1}{n}$

$= \frac{1}{n}\frac{d}{d\lambda}\sum_{k=1}^{\infty}\lambda^k = \frac{1}{n}\frac{d}{d\lambda}\left(\frac{\lambda}{1-\lambda}\right) = \frac{1}{n}\left(\frac{(1-\lambda)(1)-\lambda(-1)}{(1-\lambda)^2}\right)$

$= \frac{1}{n(1-\lambda)^2} = \frac{1}{n(\frac{1}{n})^2} = n$]

(ii) $E(X) = 1\left(\frac{1}{n}\right) + 2\left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) + \cdots + n\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\ldots\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)$

$= \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n} \cdot \frac{1}{2}n(n+1) = \frac{1}{2}(n+1)$

(iii) **1st part**

$E(X) = 1\left(\frac{1}{n}\right) + 2\left(\frac{n-1}{n}\right)\left(\frac{1}{n+1}\right) + \cdots$

$+k\left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right)\left(\frac{n+1}{n+2}\right)\ldots\left(\frac{n+k-3}{n+k-2}\right)\left(\frac{1}{n+k-1}\right) + \cdots$

and P(correct key is drawn at kth attempt)

$= \left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right)\left(\frac{n+1}{n+2}\right)\dots\left(\frac{n+k-3}{n+k-2}\right)\left(\frac{1}{n+k-1}\right)$

$= \frac{n-1}{(n+k-2)(n+k-1)}$, as required.

2nd part

$E(X) = \sum_{k=1}^{\infty} \frac{k(n-1)}{(n+k-2)(n+k-1)}$

Write $\frac{k}{(n+k-2)(n+k-1)} = \frac{A}{(n+k-2)} + \frac{B}{(n+k-1)}$

so that $k = A(n+k-1) + B(n+k-2)$

Equating coeffs of k: $1 = A + B$

Equating constant terms: $0 = A(n-1) + B(n-2)$

$\Rightarrow 0 = A(n-1) + (1-A)(n-2)$

$\Rightarrow 0 = A + n - 2$

$\Rightarrow A = 2 - n$ & $B = 1 - (2-n) = n - 1$

So $E(X) = (n-1)\sum_{k=1}^{\infty}\{\frac{2-n}{(n+k-2)} + \frac{n-1}{(n+k-1)}\}$

Writing $m_1 = n + k - 2$ & $m_2 = n + k - 1$,

$E(X) = (n-1)(2-n)\left\{\sum_{m_1=n-1}^{\infty}\frac{1}{m_1}\right\} + (n-1)^2\left\{\sum_{m_2=n}^{\infty}\frac{1}{m_1}\right\}$

$= (n-1)(2-n)\left\{\sum_{m_1=1}^{\infty}\frac{1}{m_1}\right\} + (n-1)^2\left\{\sum_{m_2=1}^{\infty}\frac{1}{m_1}\right\}$ less a finite quantity,

and so, as $\sum_{m=1}^{N}\frac{1}{m} \to \infty \; as \; N \to \infty$, $E(X)$ is infinite.