STEP 2017, P2, Q12 - Sol'n (2 pages; 10/4/24)

(i) $P(X+Y=r)=\sum_{i=0}^{r}P(X=i)P(Y=r-i)$

(as X & Y are independent, $P(X=i\ \&\ Y=r-i)$

$=P(X=i)P(Y=r-i)$)

$=\sum_{i=0}^{r}\frac{e^{-\lambda}\lambda^{i}}{i!}.\frac{e^{-\mu}\mu^{r-i}}{(r-i)!}=\frac{e^{-(\lambda+\mu)}}{r!}\sum_{i=0}^{r}\binom{r}{i}\lambda^{i}\mu^{r-i}$

$=\frac{e^{-(\lambda+\mu)}}{r!}(\lambda+\mu)^{r}$, so that $X+Y\sim Po(\lambda+\mu)$

(ii) $P(X=r\,|X+Y=k)=\frac{P(X=r\ \&\ X+Y=k)}{P(X+Y=k)}$

$=\frac{P(X=r)P(Y=k-r)}{P(X+Y=k)}$ (as X & Y are independent)

$=\frac{\left(\frac{e^{-\lambda}\lambda^{r}}{r!}\right)\left(\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}\right)}{\left(\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{k}}{k!}\right)}=\frac{\lambda^{r}\mu^{k-r}}{(\lambda+\mu)^{k}}\binom{k}{r}$

$=(\frac{\lambda}{\lambda+\mu})^{r}(\frac{\mu}{\lambda+\mu})^{k-r}\binom{k}{r}$,

so that $X\mid X+Y=k\sim B(k,\frac{\lambda}{\lambda+\mu})$

(iii) Suppose it is known that a single fish has been caught by time t. Then the Poisson parameters for Adam and Eve are $\frac{t}{T}\lambda$ & $\frac{t}{T}\mu$, respectively.

And from (ii), the number of fish caught by Adam

$\sim B\left(1,\frac{\left(\frac{t}{T}\lambda\right)}{\left(\frac{t}{T}\lambda+\left(\frac{t}{T}\mu\right)\right)}\right)$; $ie\ B(1,\frac{\lambda}{\lambda+\mu})$, so that $P(\text{1st fish has been caught by Adam}) = \frac{\lambda}{\lambda+\mu}$

(iv) [The time until the next event follows an exponential distribution, with pdf $f(x) = \lambda e^{-\lambda x}$.]

Expected time = Expected time until A&E catch their 1st fish

+P(A catches the 1st fish) × Expected time until E catches her 1st fish [from when A catches his 1st fish]

+P(E catches the 1st fish) × Expected time until A catches his 1st fish [from when E catches her 1st fish]

$= \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\left(\frac{1}{\mu}\right) + \frac{\mu}{\lambda+\mu}\left(\frac{1}{\lambda}\right)$

$= \frac{\lambda\mu+\lambda^2+\mu^2}{(\lambda+\mu)\lambda\mu}$

[This can be shown to equal $\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$, as per the official sol'n (alternatively, the 1st line of the official sol'n can be seen to equal the 1st line above).]