STEP 2017, P1, Q2 - Solution (2 pages; 6/10/19)

(i) $\frac{1}{t} \leq 1$ for $t \geq 1 \Rightarrow \int_1^x \frac{1}{t}\, dt \leq \int_1^x 1\, dt \Rightarrow [lnt]_1^x \leq [t]_1^x$

$\Rightarrow lnx \leq x - 1$, where $x \geq 1$

$\frac{1}{t} \geq 1$ for $0 < t < 1$, so if $0 < x \leq 1$, $\int_x^1 \frac{1}{t}\, dt \geq \int_x^1 1\, dt$

$\Rightarrow [lnt]_x^1 \geq [t]_x^1 \Rightarrow -lnx \geq 1 - x \Rightarrow lnx \leq x - 1$ again

(ii) $\frac{1}{t^2} \leq \frac{1}{t}$ for $t \geq 1 \Rightarrow \int_1^x \frac{1}{t^2}\, dt \leq \int_1^x \frac{1}{t}\, dt \Rightarrow \left[-\frac{1}{t}\right]_1^x \leq [lnt]_1^x$

$\Rightarrow 1 - \frac{1}{x} \leq lnx \Rightarrow lnx \geq 1 - \frac{1}{x}$ for $x \geq 1$

For $0 < t < 1$, $\frac{1}{t^2} \geq \frac{1}{t}$, so that $\int_x^1 \frac{1}{t^2}\, dt \geq \int_x^1 \frac{1}{t}\, dt$

$\Rightarrow \left[-\frac{1}{t}\right]_x^1 \geq [lnt]_x^1 \Rightarrow \frac{1}{x} - 1 \geq -lnx$

$\Rightarrow lnx \geq 1 - \frac{1}{x}$ when $0 < x < 1$

So $lnx \geq 1 - \frac{1}{x}$ for $x > 0$

(iii) $\int 1.lnx\, dx = xlnx - \int x.\frac{1}{x}\, dx = xlnx - x\ (+c)$

From (*), if $y \geq 1$, $\int_1^y lnx\, dx \leq \int_1^y x - 1\ dx$

$\Rightarrow [\, xlnx - x]_1^y \leq [\frac{1}{2}x^2 - x]_1^y$

$\Rightarrow ylny - y + 1 \leq \frac{1}{2}y^2 - y - \frac{1}{2} + 1$

$\Rightarrow ylny \leq \frac{1}{2}y^2 - \frac{1}{2}$

$\Rightarrow ylny \leq \frac{1}{2}(y - 1)(y + 1)$

$\Rightarrow \frac{lny}{y-1} \leq \frac{y+1}{2y}$, provided $y \neq 1$

If $0 < y < 1$, (*)$\Rightarrow \int_y^1 lnx\ dx \leq \int_y^1 x - 1\ \ dx$

$\Rightarrow [\ xlnx - x]_y^1 \leq [\frac{1}{2}x^2 - x]_y^1$

$\Rightarrow -1 - ylny + y \leq \frac{1}{2} - 1 - \frac{1}{2}y^2 + y$

$\Rightarrow -ylny \leq \frac{1}{2} - \frac{1}{2}y^2$

$\Rightarrow -ylny \leq \frac{1}{2}(1-y)(1+y)$

$\Rightarrow \frac{lny}{y-1} \leq \frac{y+1}{2y}$, as $1 - y > 0$

So $\frac{lny}{y-1} \leq \frac{y+1}{2y}$ for $y > 0$ & $y \neq 1$

To establish the left-hand inequality:

From (**), if $y \geq 1$, $\int_1^y lnx\ dx \geq \int_1^y 1 - \frac{1}{x}\ \ dx$

$\Rightarrow [xlnx - x]_1^y \geq [x - lnx]_1^y$

$\Rightarrow ylny - y + 1 \geq y - lny - 1$

$\Rightarrow lny(y+1) \geq 2y - 2$

$\Rightarrow \frac{lny}{y-1} \geq \frac{2}{y+1}$, provided $y \neq 1$

If $0 < y < 1$, (**)$\Rightarrow \int_y^1 lnx\ dx \geq \int_y^1 1 - \frac{1}{x}\ \ dx$

$\Rightarrow [xlnx - x]_y^1 \geq [x - lnx]_y^1$

$\Rightarrow -1 - ylny + y \geq 1 - y + lny$

$\Rightarrow lny(1 + y) \leq -2 + 2y$

$\Rightarrow \frac{lny}{y-1} \geq \frac{2}{y+1}$, as $y - 1 < 0$

So $\frac{lny}{y-1} \geq \frac{2}{y+1}$ for $y > 0$ (and $y \neq 1$)

Note: In the official mark scheme, in the alternative method for (ii), the last 3 lines should say:

"Therefore, ... the LHS [lnx] grows more rapidly for $\boldsymbol{x > 1}$ and the RHS [$1 - \frac{1}{x}$] grows more rapidly for $\boldsymbol{x < 1}$, the inequality is true."

[as can be seen by sketching the 2 curves]