STEP 2017, P1, Q1 - Solution (2 pages; 4/10/19)

(i) Let $u = xsinx + cosx$,

so that $du = (sinx + xcosx - sinx)dx = xcosx\, dx$

Then $\int \frac{x}{xtanx+1}\, dx = \int \frac{xcosx}{xsinx+cosx}\, dx$

$= \int \frac{1}{u}\, du = \ln|xsinx + cosx| + C$

$I = \int \frac{x}{xcotx-1}\, dx = \int \frac{xsinx}{xcosx-sinx}\, dx$

$\frac{d}{dx}(xcosx - sinx) = cosx - xsinx - cosx = -xsinx$

So let $u = xcosx - sinx$,

so that $I = -\int \frac{1}{u}\, du = -ln|xcosx - sinx| + D$

(ii) [If necessary, try the simplest possible substitution. See comments later on.]

$\frac{d}{dx}(xsec^2x - tanx) = sec^2x + x\frac{d}{dx}(cosx)^{-2} - sec^2x$

$= -2x(cosx)^{-3}(-sinx) = 2xtanxsec^2x$

So let $u = xsec^2x - tanx$,

so that $I = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}ln|\, xsec^2x - tanx| + E$

[As the first part of (ii) turned out to be straightforward, it is likely that it is needed in some way for the 2nd part of (ii).]

[The simplest approach for the 2nd part of (ii) is to convert it to the form $\int \frac{f\prime(x)}{(f(x))^2} dx$.]

As $\frac{xsinxcosx}{(x-sinxcosx)^2} = \frac{xtanxsec^2x}{(xsec^2x-tanx)^2}$,

let $u = xsec^2x - tanx$ again,

so that the integral $= \frac{1}{2}\int \frac{2xtanxsec^2x}{(xsec^2x-tanx)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\ du$

$= -\frac{1}{u} + F = -\frac{1}{2(xsec^2x-tanx)} + F$