STEP 2017, P1, Q13 - Solution (4 pages; 18/2/21)

1st part

$s_1 = 0$

2nd part

$\mathrm{t_r} = P\big((\mathrm{r}-1)\text{st slice is used for 1st half of sandwich}\big)$

$\times$ (P(rth slice is used for toast|

$(\mathrm{r}-1)$st is used for 1st half of sandwich)

$+\mathrm{s_{r-1}}$P(rth slice is used for toast|

$(\mathrm{r}-1)$st is used for 2nd half of sandwich)

$+\ \mathrm{t_{r-1}}$P(rth slice is used for toast|

$(\mathrm{r}-1)$st is used for toast)

$= P\big((\mathrm{r}-1)\text{st slice is used for 1st half of sandwich}\big) \times 0$

$+\mathrm{s_{r-1}}p + \mathrm{t_{r-1}}p$

$= (\mathrm{s_{r-1}} + \mathrm{t_{r-1}})p$ for $2 \leq r \leq n-1$

($r \neq 1$, as $\mathrm{s_0}$ and $\mathrm{t_0}$ aren't defined; for $r = n$, the probabilities are different)

$s_r = P((\mathrm{r}-1)\text{st slice is used for 1st half of sandwich})$

$= 1 - (\mathrm{s_{r-1}} + \mathrm{t_{r-1}})$ for $2 \leq r \leq n$

(again, $r \neq 1$, as $\mathrm{s_0}$ and $\mathrm{t_0}$ aren't defined)

3rd part

Substituting from the 1st eq'n into the 2nd: $s_r = 1 - \frac{t_r}{p}$

Hence $s_{r-1} = 1 - \frac{t_{r-1}}{p}$ for $2 \le r - 1 \le n - 1$; ie for $3 \le r \le n$,

so that $t_{r-1} = p(1 - s_{r-1})$

Then, substituting into the 2nd eq'n:

$s_r = 1 - s_{r-1} - p(1 - s_{r-1})$

$= (1 - p) - s_{r-1}(1 - p) = q(1 - s_{r-1})$ for $3 \le r \le n - 1$,

as the 1st eq'n was limited to $r \le n - 1$

We also need to show that $s_2 = q(1 - s_1) = q$,

but this follows from the fact that the 2nd side of the sandwich will be made straightaway if the 1st has just been made (and there is probability q that a sandwich is started with the 1st slice of bread).

So $s_r = q(1 - s_{r-1})$ for $2 \le r \le n - 1$, as required.

4th part

Proof by induction:

$s_1 = 0$ and $\frac{q+(-q)^1}{1+q} = 0$; so the result holds for $n = 1$

Assume that it holds for $n = k$, so that $s_k = \frac{q+(-q)^k}{1+q}$

Then $s_{k+1} = q(1 - s_k)$ for $2 \le k + 1 \le n - 1$; ie $1 \le k \le n - 2$

$= q - \frac{q[q+(-q)^k]}{1+q}$

$= \frac{1}{1+q}\{q(1 + q) - q^2 + (-q)^{k+1}\}$

$= \frac{q+(-q)^{k+1}}{1+q}$

Thus, if the result is true for $n = k$, it will be true for $n = k + 1$, provided that $1 \leq k \leq n - 2$

As the result is true for $n = 1$, it follows by the principle of induction that it will be true for $1 \leq k \leq n - 1$, as required.

5th part

From the start of the 3rd part, $s_r = 1 - \frac{t_r}{p}$ (for $2 \leq r \leq n - 1$, as it is based on both of the given eq'ns),

so that $t_r = p(1 - s_r) = \frac{p}{1+q}\{(1 + q) - [q + (-q)^r]\}$

$= \frac{p}{1+q}\{1 - (-q)^r\}$ for $2 \leq r \leq n - 1$

Also, $t_1 = p$ and $\frac{p}{1+q}\{1 - (-q)^1\} = p$,

so that $t_r = \frac{p}{1+q}\{1 - (-q)^r\}$ for $1 \leq r \leq n - 1$

6th part

From the 2nd of the given eq'ns,

$s_r = 1 - (\mathrm{s_{r-1}} + \mathrm{t_{r-1}})$ for $2 \leq r \leq n$,

so that $s_n = 1 - (s_{n-1} + t_{n-1})$

$= 1 - \frac{q+(-q)^{n-1}}{1+q} - \frac{p}{1+q}\{1 - (-q)^{n-1}\}$

$= \frac{1}{1+q}\{(1 + q) - [q + (-q)^{n-1}] - p[1 - (-q)^{n-1}]\}$

$= \frac{1}{1+q}\{q - q(-q)^{n-1}\}$

$= \frac{1}{1+q}\{q + (-q)^n\}$

Also, $t_n = P\big((\mathrm{n}-1)\text{st slice is used for 1st half of sandwich}\big)$

$\times$ (P(nth slice is used for toast|

(n − 1)st is used for 1st half of sandwich)

$+\mathrm{s}_{\mathrm{n}-1}$P(nth slice is used for toast|

(n − 1)st is used for 2nd half of sandwich)

$+\ \mathrm{t}_{\mathrm{n}-1}$P(nth slice is used for toast|

(n − 1)st is used for toast)

$= P\big((\mathrm{n}-1)\text{st slice is used for 1st half of sandwich}\big) \times 0$

$+(\mathrm{s}_{\mathrm{n}-1} \times 1) + (\mathrm{t}_{\mathrm{n}-1} \times 1)$

$= \mathrm{s}_{\mathrm{n}-1} + \mathrm{t}_{\mathrm{n}-1}$

$= \frac{1}{1+q}\{q + (-q)^{n-1}\} + \frac{p}{1+q}\{1 - (-q)^{n-1}\}$

$= \frac{1}{1+q}\{(q + p) + (1 - p)(-q)^{n-1}\}$

$= \frac{1}{1+q}\{1 + q(-q)^{n-1}\}$

$= \frac{1}{1+q}\{1 - (-q)^{n}\}$