STEP 2016, Paper 3, Q8 – Solution (3 pages; 6/5/19)

(i) Writing x instead of $-x$ (as equation is valid for all x),

$f(-x) + (1 - [-x])f(-[-x]) = [-x]^2$,

and so $f(-x) + (1 + x)f(x) = x^2$, as required. (A)

The original equation was $f(x) + (1 - x)f(-x) = x^2$ (B)

Subst. for $f(-x)$ from (A) into (B) gives

$f(x) + (1 - x)[x^2 - (1 + x)f(x)] = x^2$ (C)

$\Rightarrow f(x)[1 - (1 - x^2)] = x^2 - (1 - x)x^2$,

so that $f(x)x^2 = x^3$

$\Rightarrow f(x) = x$ unless $x = 0$

When $x = 0$, (B) $\Rightarrow f(0) + f(0) = 0$,

so that $f(0) = 0$.

Thus $f(x) = x$ for all x.

Verification: Subst. into (B) gives:

LHS $= x + (1 - x)(-x) = x^2 =$ RHS

(ii) $K(K(x)) = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$, as required

Given: $g(x) + xg\left(\frac{x+1}{x-1}\right) = x$ (D)

[To eliminate $g\left(\frac{x+1}{x-1}\right)$, create another equation involving $g\left(\frac{x+1}{x-1}\right)$, by replacing x with $\frac{x+1}{x-1}$ in (D), and hope that something fortuitous will happen.]

Replacing x with $\frac{x+1}{x-1}$ in (D),

$$g\left(\frac{x+1}{x-1}\right)+\frac{x+1}{x-1}g\left(\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}\right)=\frac{x+1}{x-1}$$

As $g\left(\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}\right)=g\Big(K\big(K(x)\big)\Big)=g(x)$,

this becomes $g\left(\frac{x+1}{x-1}\right)+\frac{x+1}{x-1}g(x)=\frac{x+1}{x-1}$ (E)

(D) and (E) can be written as

$g(x)+xg(K(x))=x$ (D')

and $g(K(x))+K(x)g(x)=K(x)$ (E')

Eliminating $g(K(x))$ by substituting from (E') into (D'),

$g(x)+x[K(x)-K(x)g(x)]=x$

$\Rightarrow g(x)\{1-xK(x)\}=x[1-K(x)]$

$\Rightarrow g(x)=\frac{x(1-\frac{x+1}{x-1})}{1-x\left(\frac{x+1}{x-1}\right)}=\frac{x(x-1-x-1)}{x-1-x(x+1)}$

$=\frac{-2x}{-1-x^2}=\frac{2x}{x^2+1}$, as required.

(iii) Given: $h(x)+h(y)=1-x-y$, where $y=\frac{1}{1-x}$ (F)

[As for $g(x)$, aim to eliminate $h(y)$ by creating another equation involving $h(y)$, by replacing x with $\frac{1}{1-x}$ in (F)]

Replacing x with $\frac{1}{1-x}$ in (F),

$$h(y)+h\left(\frac{1}{1-\left(\frac{1}{1-x}\right)}\right)=1-\frac{1}{1-x}-\frac{1}{1-\left(\frac{1}{1-x}\right)}$$

$\Rightarrow h(y) + h\left(\frac{1-x}{1-x-1}\right) = 1 - \frac{1}{1-x} - \frac{1-x}{1-x-1}$

$\Rightarrow h(y) + h\left(\frac{x-1}{x}\right) = 1 - \frac{1}{1-x} - \frac{x-1}{x}$

or $h(y) + h(z) = 1 - y - z$, where $z = \frac{x-1}{x}$ (G)

[Now aim to eliminate $h(z)$ by creating another equation involving $h(z)$, by replacing x with $\frac{x-1}{x}$ in (F), say]

Replacing x with $\frac{x-1}{x}$ in (F),

$h(z) + h\left(\frac{1}{1-\left(\frac{x-1}{x}\right)}\right) = 1 - \frac{x-1}{x} - \frac{1}{1-\left(\frac{x-1}{x}\right)}$

$\Rightarrow h(z) + h\left(\frac{x}{x-x+1}\right) = 1 - z - \frac{x}{x-x+1}$

$\Rightarrow h(z) + h(x) = 1 - z - x$ (H)

So $h(x) + h(y) = 1 - x - y$ (F)

$h(y) + h(z) = 1 - y - z$ (G)

$h(z) + h(x) = 1 - z - x$ (H)

And $(F) - (G) + (H) \Rightarrow 2h(x)$

$= (1 - x - y) - (1 - y - z) + (1 - z - x)$

$= 1 - 2x,$

so that $h(x) = \frac{1}{2} - x$