**STEP 2016, Paper 3, Q2 – Solution** (2 pages; 6/5/19)

(i) Gradient of PQ $= \frac{2ap-2aq}{ap^2-aq^2} = \frac{2(p-q)}{p^2-q^2} = \frac{2}{p+q}$

Also, tangent to curve at $(at^2, 2at)$ has gradient

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$, so that gradient of normal at Q is $-q$

Thus $\frac{2}{p+q} = -q$ and so $2 = -qp - q^2$ and $q^2 + qp + 2 = 0$, as required.

(ii) By symmetry, the point can be expected to lie on the $x$-axis.

Equation of QR is $\frac{y-2aq}{x-aq^2} = \frac{2ar-2aq}{ar^2-aq^2} = \frac{2}{r+q}$ (A)

When $y = 0$, $-aq(r+q) = x - aq^2$,

so that $x = -aqr$

Also, from (i), $q^2 + qp + 2 = 0$ and similarly for $r$;

ie $q$ and $r$ are solutions of $x^2 + px + 2 = 0$ ,

so that $qr = 2$, and hence the required point is $(-2a, 0)$.

(iii) By symmetry, the line can be expected to be parallel to the $y$-axis.

OP has equation $y = x\left(\frac{2ap}{ap^2}\right) = \frac{2x}{p}$ (B)

Using (A) to eliminate $y$ gives

$\frac{2x}{p} = 2aq + \frac{2(x-aq^2)}{r+q}$

$\Rightarrow x(r+q) = aqp(r+q) + (x-aq^2)p$

$\Rightarrow x(r+q-p) = apqr$

$\Rightarrow x = \frac{apqr}{r+q-p}$

Then, as $q$ and $r$ are solutions of $x^2 + px + 2 = 0$ ,

$qr = 2$ and $r + q = -p$,

so that $x = \frac{2ap}{-2p} = -a$

ie the required line is $x = -a$

From (B), T is $(-a, -\frac{2a}{p})$, so that the distance from the $x$-axis to T is $\frac{2a}{p}$

To show that $\frac{2a}{p} < \frac{a}{\sqrt{2}}$; ie that $p > 2\sqrt{2}$ (assuming $p > 0$; if $p < 0$, then $\frac{2a}{p} < \frac{a}{\sqrt{2}}$ automatically, as $a > 0$:

In order for Q and R to be distinct, the discriminant of

$x^2 + px + 2 = 0$ must be positive; so that $p^2 - 8 > 0$; ie $p > 2\sqrt{2}$, as required.