STEP 2016, Paper 3, Q1 – Solution (2 pages; 6/5/19)

(i) $x + a = \sqrt{b - a^2}\,tanu \Rightarrow dx = \sqrt{b - a^2}\,sec^2u\,du$

and $x^2 + 2ax + b = (x + a)^2 + b - a^2 = (b - a^2)tan^2u + b - a^2$

$= (b - a^2)sec^2u$

Also, $x = \pm\infty \Rightarrow u = \pm\frac{\pi}{2}$

So $I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\,sec^2u\,du}{(b-a^2)sec^2u} = \frac{1}{\sqrt{b-a^2}}[u]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{b-a^2}}$

(ii) With the same substitution, $I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\,sec^2u\,du}{(b-a^2)^n sec^{2n}u}$

$= (b - a^2)^{\frac{1}{2}-n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2n-2}u\,du$

and $I_{n+1} = (b - a^2)^{\frac{1}{2}-(n+1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2(n+1)-2}u\,du$

Result to prove: $2n(b - a^2)I_{n+1} - (2n - 1)I_n = 0$ (*)

Writing $J_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2n}u\,du,$

LHS of (*) $= 2n(b - a^2)^{\frac{1}{2}-n}J_n - (2n - 1)(b - a^2)^{\frac{1}{2}-n}J_{n-1}$ (A)

Then $J_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2n}u\,du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2n-1}u\,cosu\,du$

$= [cos^{2n-1}sinu]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2n - 1)cos^{2n-2}u(-sinu)sinu\,du$,

by Parts

$= 0 + (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{2n-2}u(1-cos^2u)\,du$

$= (2n-1)(J_{n-1} - J_n)$,

so that $2nJ_n = (2n-1)J_{n-1}$

Then $(A) = (b-a^2)^{\frac{1}{2}-n}[2nJ_n - (2n-1)J_{n-1}] = 0$, as required.

(iii) $I_1 = \frac{\pi}{2^0(b-a^2)^{\frac{1}{2}}}\binom{0}{0} = \frac{\pi}{\sqrt{b-a^2}}$, so that the result is true for $n = 1$

Assume that the result is true for $n = k$, so that

$I_k = \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}}\binom{2k-2}{k-1}$

Result to prove: $I_{k+1} = \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}}\binom{2k}{k}$

From (ii), $I_{k+1} = \frac{(2k-1)I_k}{2k(b-a^2)} = \frac{(2k-1)}{2k(b-a^2)}\frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}}\binom{2k-2}{k-1}$

$= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}}\{\frac{(2k-1)2^2}{2k}\binom{2k-2}{k-1}\}$ (B)

And $\frac{(2k-1)2^2}{2k}\binom{2k-2}{k-1} = \frac{2(2k-1)}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{(2k)!}{k^2(k-1)!(k-1)!}$

$= \frac{(2k)!}{k!k!} = \binom{2k}{k}$, so that (B) $= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}}\binom{2k}{k}$, as required.

Thus, if the result is true for $n = k$, then it is true for $n = k+1$. As it is true for $n = 1$, it is therefore true for $n = 2,3, \ldots$, and so for all positive integers, by the principle of induction.