STEP 2016, Paper 3, Q11 – Solution (3 pages; 20/2/21)

(i) N2L $\Rightarrow D(t) - k\dot{x} = m\ddot{x}$ (1)

where D is the driving force of the engine, and $P = D(t)\dot{x}$

When $\ddot{x} = 0, \dot{x} = 4U$, so that $\frac{P}{4U} - k(4U) = 0$, and so $k = \frac{P}{16U^2}$

(1) can be rewritten as $\frac{P}{v} - \frac{Pv}{16U^2} = m\frac{dv}{dt} = m\frac{dv}{dx} . \frac{dx}{dt} = m\frac{dv}{dx}\ v$

$\Rightarrow \int_0^{X_1} dx = \frac{m}{P}\int_U^{2U} \frac{v}{\left(\frac{1}{v} - \frac{v}{16U^2}\right)} dv$

$\Rightarrow X_1 = \frac{16mU^2}{P}\int_U^{2U} \frac{v^2}{16U^2 - v^2} dv$

$= \frac{16mU^2}{P}\{\int_U^{2U} \frac{v^2 - 16U^2}{16U^2 - v^2} dv + \int_U^{2U} \frac{16U^2}{16U^2 - v^2} dv\}$

$= \frac{16mU^2}{P}\{(U - 2U) + 16U^2\int_U^{2U} \frac{1}{(4U - v)(4U + v)} dv\}$

$= \frac{16mU^2}{P}\{-U + \frac{16U^2}{8U}\int_U^{2U} \frac{1}{4U - v} + \frac{1}{4U + v} dv\}$

$= \frac{16mU^2}{P}\{-U + 2U[-ln|4U - v| + ln|4U + v|]_U^{2U}$

$= \frac{16mU^2}{P}\{-U + 2U(-\ln(2U) + \ln(6U) + \ln(3U) - \ln(5U))\}$

$= \frac{16mU^3}{P}\{-1 + 2\ln\left(\frac{(6U)(3U)}{(2U)(5U)}\right)\}$

$= \frac{1}{\lambda}(-1 + 2\ln\left(\frac{9}{5}\right))$

And so $\lambda X_1 = 2\ln\left(\frac{9}{5}\right) - 1$, as required.

(ii) N2L $\Rightarrow D(t) - k(\dot{x})^2 = m\ddot{x}$ (2)

and $P = D(t)\dot{x}$

When $\ddot{x} = 0, \dot{x} = 4U$, so that $\frac{P}{4U} - k(4U)^2 = 0$, and so $k = \frac{P}{64U^3}$

(2) can be rewritten as $\frac{P}{v} - \frac{Pv^2}{64U^3} = m\frac{dv}{dx}\ v$

$\Rightarrow \int_0^{X_2} dx = \frac{m}{P}\int_U^{2U} \frac{v}{\left(\frac{1}{v} - \frac{v^2}{64U^3}\right)} dv$

$\Rightarrow X_2 = \frac{64mU^3}{P}\int_U^{2U} \frac{v^2}{64U^3 - v^3} dv$

$= \frac{64mU^3}{-3P}\int_U^{2U} \frac{-3v^2}{64U^3 - v^3} dv$

$= \frac{64mU^3}{-3P}[ln|64U^3 - v^3|]_U^{2U}$

$= \frac{64mU^3}{-3P}(\ln(56U^3) - \ln(63U^3))$

$= \frac{64mU^3}{-3P}\ln\left(\frac{56U^3}{63U^3}\right)$

$= -\frac{4}{3\lambda}\ln\left(\frac{8}{9}\right)$

And so $\lambda X_2 = \frac{4}{3}\ln\left(\frac{9}{8}\right)$, as required.

(iii) Consider $\lambda X_2 - \lambda X_1 = \frac{4}{3}\ln\left(\frac{9}{8}\right) - 2\ln\left(\frac{9}{5}\right) + 1$

$= \frac{4}{3}ln9 - \frac{4}{3}ln8 - 2ln9 + 2ln5 + 1$

$= 2ln5 + 1 - \frac{2}{3}ln9 - \frac{4}{3}ln8$

$= 2ln5 + 1 - \frac{4}{3}ln3 - \frac{4}{3}ln8$

$= 2ln5 + 1 - \frac{4}{3}ln24$ (A)

Now (A)$> 2(1.60) + 1 - \frac{4}{3}(3.18) = 4.20 - 4(1.06)$

(which doesn't help)

But (A)$< 2(1.61) + 1 - \frac{4}{3}(3.17) = 4.22 - 4(1.05 + \frac{2}{300})$

$= 0.02 - \frac{8}{300} = \frac{-2}{300} < 0$

So $\lambda X_2 - \lambda X_1 < 0$, and X_1 is therefore the larger of the two.